Implied liquidity: towards stochastic liquidity modelling and liquidity trading

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Abstract: In this paper the authors introduce the new concept of implied liquidity based on the recent developed two-way price theory (conic finance). Implied liquidity isolates and quantifies liquidity risk in financial markets. It is shown on real market option data on the major US indices how liquidity dried up in the troubled year end of 2008. These investigations open the door to stochastic liquidity modelling, liquidity derivatives and liquidity trading.

Keywords: conic finance; stochastic liquidity; liquidity trading; liquidity derivatives.

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1 Introduction

In business, economics or investment, market liquidity is an important quantity. In portfolio analysis and management it is therefore a key ingredient to consider for risk-management purposes. It reflects the asset's ability to be sold without causing a significant movement in the price and with minimum loss of value. Liquidity goes hand in hand with bid and ask spreads; highly liquid assets have a small spread; illiquid assets have a high spread. The essential characteristic of a liquid market is that there are ready and willing buyers and sellers at all times. Some products are more liquid than other investments. Clearly, this depends on the asset in question. For example, Dow Jones index components are obviously much more liquid than real estate.

Investors know that other agents in the market will be less willing to buy an illiquid asset than an alternative liquid asset offering a similar return. Hence, prudent investors rationally price illiquid assets at a discount in such a way that the enhanced return offsets the enhanced liquidity risk. Hence to be prudent it has been argued that if one buys an asset at its ask price, it should be booked on the basis of its bid price (Madan and Schoutens, 2011a) (and hence a transaction immediately produces an accounting loss, namely the bid-ask spread).

Huge investment portfolios are subject to systematic and structural liquidity risk. In times of crisis, liquidity dries up and one can not easily unwind positions near theoretical prices. Fire-sale transactions are typically at much lower prices, due to huge bid-ask spreads at such moments.

However, it is very difficult to measure liquidity in an isolated manner. Bid-ask spread is a good indication, but it moves around in a non-linear manner if volatility or spot moves, even if the underlying liquidity of the asset does not actually change. The main contribution of this paper is the introduction of the concept of implied liquidity. Implied liquidity allows one to quantify in a unique and fundamental founded way the liquidity risk in financial markets. The main advantages of such a new measure is that it makes comparison over time, products and asset classes, possible. Like implied volatility allows investors to assess the market risk (measures as standard deviation of the return distribution) of underlying assets, implied liquidity allows investors to measure in a similar market implied fashion the liquidity level of positions.

In this paper, we will, by making use of conic finance theory, introduce the concepts of a two-price market (bid and ask prices). The theory essentially models the liquidity of a given asset at a given point in time. We elaborate on the calculation of bid and ask prices for vanillas in the Black-Scholes world. Going beyond the implied volatility, which reflects the mid-price of a vanilla, we now also introduce an implied liquidity parameter. It is a unit-less quantity that makes comparison of liquidity across different assets and markets straightforward. We calculate these implied liquidity parameters over different strikes and maturities and hence come to implied liquidity surfaces. Next, we study the behaviour of these quantities over time. More precisely we calculate implied liquidity for the ATM calls on US indices over time. We clearly see the effect during the credit crisis; the implied liquidity parameter spiked up during the weeks after Lehman collapsed, indicating a clear drying up of liquidity in major vanilla markets.

Finally, we are tempted to introduce some possible liquidity derivative contracts, which could serve as potential hedge instruments against the drying up of liquidity, and we illustrate their use.

2 Conic finance bid and ask pricing

In this section, we summarise the basic conic finance techniques used. For more background and some applications, see Cherny and Madan (2009), Cherny and Madan (2010), Eberlein and Madan (2009), Eberlein et al. (2010), Madan and Schoutens

(2011a) and Madan and Schoutens (2011b). We will discuss distorted expectation, acceptability and bid-ask pricing.

In this paper, we will make use of a distortion function (suggested by Cherny and Madan, 2009) from the minmaxvar family parameterised as given in equation (1) by one parameter lambda $\lambda \geqslant 0$

$$\Phi(u;\lambda) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\lambda}. \tag{1}$$

Securities are traded in their own markets and we model different markets using different levels of lambda to reflect the different liquidity situations in these markets.

More precisely, we use a distorted expectation to calculate bid and ask prices. The prices arise from the theory of acceptability. We say that a risk X is acceptable (notation: $X \in \mathcal{A}$) if

$E_Q[X] \geq 0$ for all measures Q in a convex set $\mathcal{M}$.

The convex set $\mathcal{M}$ contains the supporting measures, which can be seen as kind of test-measures under which the cash-flow needs to have positive expectation to deliver acceptability. Under a larger set $\mathcal{M}$, one has a smaller set of acceptable risks, because there are, underlying, more tests to be passed. Operational cones were defined by Cherny and Madan (2009) and depend solely on the distribution function $G(x)$ of X and a distortion function Φ. $X \in \mathcal{A}$ if the distorted expectation is non-negative.

More precisely, the distorted expectation of a random variable X with distribution function $G(x)$ relative to the distortion function Φ (we use the one given in equation (1), but other distortion functions are also possible), is defined as

$$de(X;\lambda) = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(G(x);\lambda). \tag{2}$$

Note that if $\lambda = 0$, $\Phi(u;0) = u$ and hence $de(X;0)$ is the ordinary expectation.

The ask price of payoff X is determined as

$$ask(X) = -\exp(-rT)de(-X;\lambda).$$

This formula is derived by noting that the cash-flow of selling X at its ask price is acceptable in the relevant market: $ask(X) - X \in \mathcal{A}$.

Similarly, the bid price of payoff X is determined as

$$bid(X) = \exp(-rT)de(X;\lambda).$$

Here the cash-flow of buying X at its bid price is acceptable in the relevant market: $X - bid(X) \in \mathcal{A}$.

One can prove that the bid and ask prices of a positive contingent claim X with distribution function $G(x)$ can be calculated as:

$$bid(X) = \exp(-rT)\int_{0}^{+\infty} x \mathrm{d}\Phi(G(x);\lambda), \tag{3}$$

$$ask(X) = \exp(-rT)\int_{-\infty}^{0} (-x) \mathrm{d}\Phi(1 - G(-x);\lambda). \tag{4}$$

Suppose we have a bid and ask price for a European call. We then can calculate the mid price of that call option, as the average of the bid and ask prices. Using this mid price we calculate the implied Black-Scholes volatility. Next, we can calculate an implied λ such that conic bid and ask prices (using the implied vol as parameter) are as best as possible matched with market prices.

Under the Black-Scholes framework, this comes down to the following calculations for a European call option with strike K and maturity T.

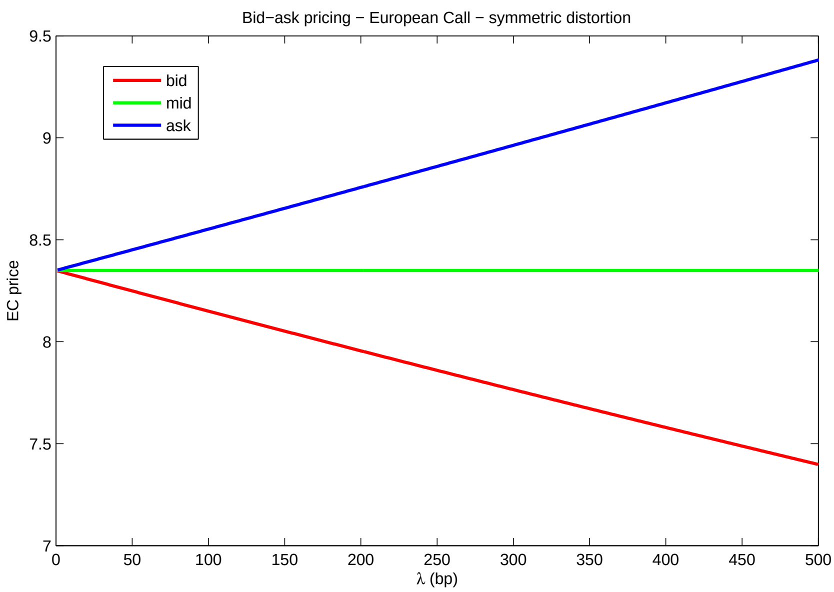
Because $P((S_T - K)^+ \geq x) = P(S_T \geq K + x)$ for $x \geq 0$, the distribution function value in point x of the call payoff, is nothing else than one minus the probability of finishing above $K + x$. Using the usual interpretation of the so-called $N(d_2)$-term of the Black-Scholes formula, we have

$$G(x) = 1 - \mathrm{N}\left(\frac{\log(S_0/(K+x)) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}\right), \qquad x \geq 0$$

where N is the cumulative distribution function of the standard normal law, σ is the implied vol determined on the basis of the mid price. For $x < 0$, $G(x) = 0$, since the payoff of a call option is a non-negative random variable. The above close-form solution for $G(x)$ in combination with equations (3) and (4) give rise to very fast and accurate calculations of the bid and ask prices.

In Figure 1, one sees the bid, mid and ask prices for a range of λ's. Values are graphed for a one year ATM call option under a 20% volatility.

Figure 1 European call – bid and ask prices



3 Implied liquidity

3.1 Definition and illustration during crisis

We will call the parameter, $\lambda > 0$, fitting the bid-ask spread around the mid price best, *the implied liquidity parameter*. As it can be seen from Figure 1, the smaller the implied liquidity parameter the more liquid the underlying and the smaller the bid-ask spread. In the extremal case where the implied liquidity parameter equals 0, the bid price coincides with the ask price, and we do work again under the one-price framework.

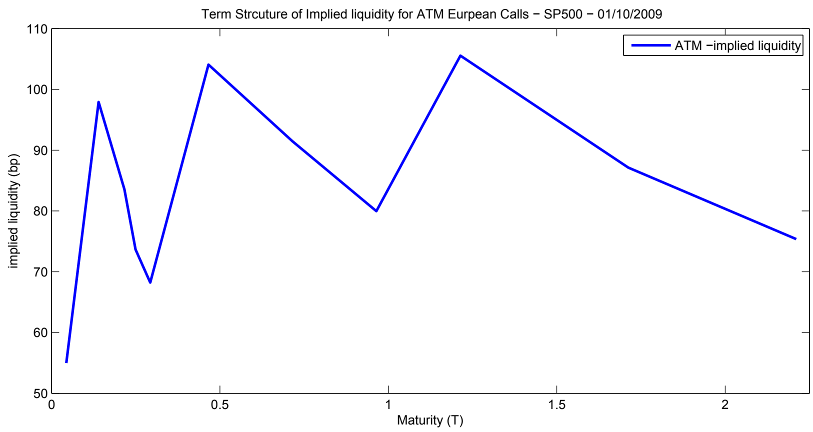
We realise that a potential criticism to the above is the model-dependency of the derived implied liquidity parameter λ. However, we note also that the same holds for the nevertheless widely successful implied volatility parameter. Both are intrinsically calculated under the Black-Scholes setting.

We note that in a fixed market with no movement in the cone of acceptable risks and hence no change in liquidity as the market is then fixed, the bid ask spread can move around non-linearly with maturity and or volatility. So the spread can move in a constant market with no change in liquidity and therefore spread itself is not a perfect measure of liquidity. Implied liquidity can overcome this criticism.

A change in lambda is not just increasing the bid-ask spread but has a direct and significant impact on liquidity, because for any proposed trade on the part of a market participant, e.g., a sell order at a fixed percentage above the bid, will have a longer waiting time to be executed if lambda is increased and hence the bid drops, alternatively the price impact of an immediate sale is enhanced by the drop in the bid. Similar argument holds for a buy order at a fixed percentage below the ask. Therefore, a change in lambda is synonymous with a change in liquidity.

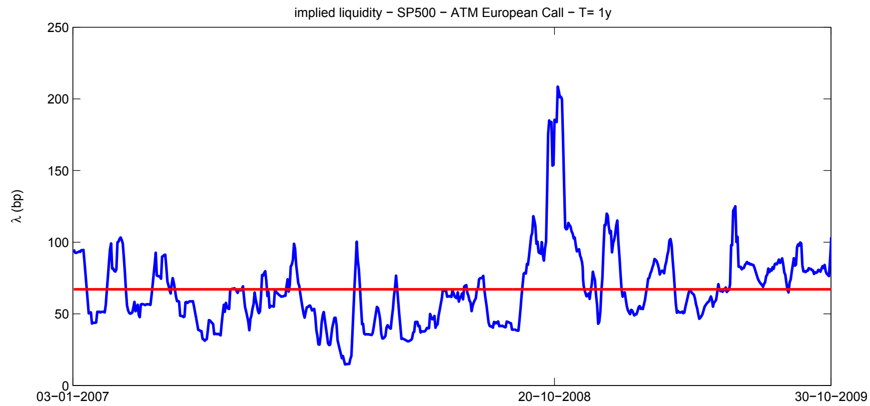
As a first illustration, we have calculated the implied liquidity parameter for ATM European calls on the SP500 over maturity on the 1st of October 2009. In Figure 2, we observe a slightly upsloping curve on average for increasing maturities; some maturities are clearly more liquid than others. In the example T=0.043, 0.293, 0.96 and 2.21 years are the most liquid ones.

Figure 2 Implied liquidity parameter for ATM European calls on the SP500 over maturity on the 1st of October 2009



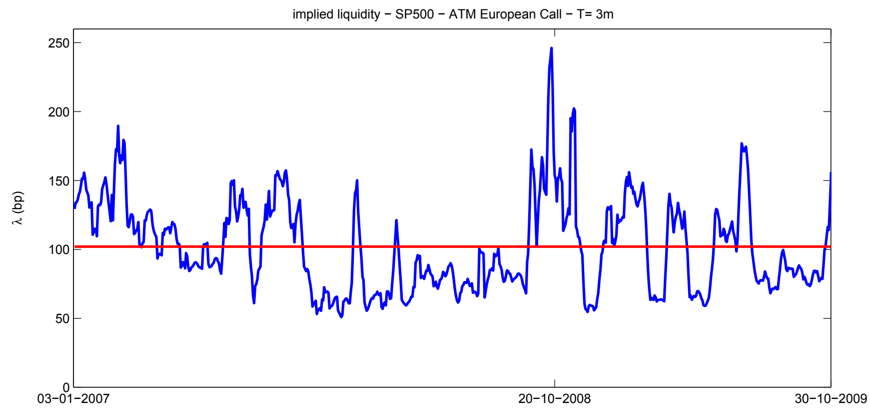
In Figure 3 we graph, the implied liquidity parameter for ATM European calls with maturity (the closed to) 1 year over time for the SP500. We clearly see that liquidity is non-constant over time and appears to exhibit a mean-reverting behaviour. The period ranges from the 3rd of January 2007 until the 30th of October 2009. We have estimated the (long run) average of the implied liquidity of the dataset and over the period of investigation this equals 67.11 bp. The highest value for the implied liquidity parameter was on the 20th of October 2008. Around that day (and the week-end before) several European banks were rescued by government interventions. The graph indicates that implied liquidity behaves in a stochastic manner and apparently has a mean-reverting nature.

Figure 3 The implied liquidity parameter for ATM European calls with maturity (the closed to) one year over time for the SP500



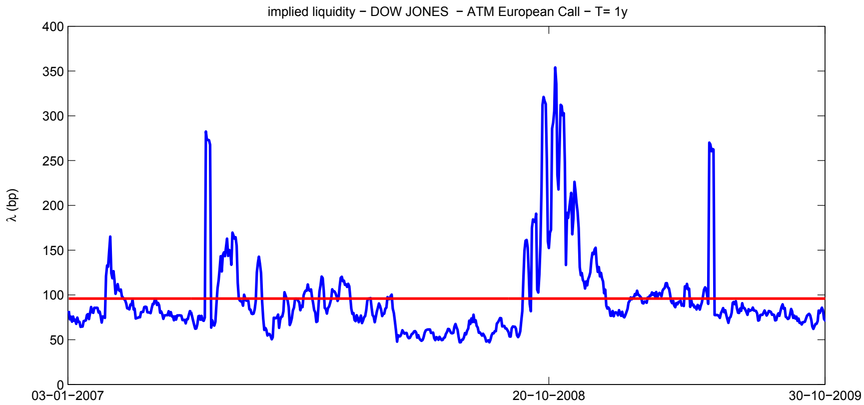
A similar graph can be found in Figure 4 for the three months to maturity ATM European call on the SP500.

Figure 4 Implied liquidity for 3 m ATM EC on SP500



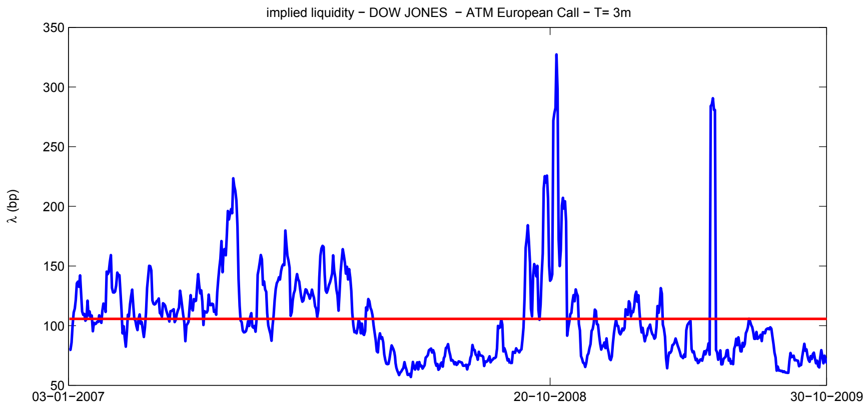
In Figure 5 we graph, the implied liquidity parameter for the ATM European call with maturity (the closed to) one year over time for the Dow Jones index. The (long run) average of the implied liquidity of the data set over the period of investigation equals 95.90 bp. The implied liquidity parameter spikes in October 2008 to around 350 bp. Liquidity is hence on average and as well in distress situations lower than on the SP500.

Figure 5 The implied liquidity parameter for the ATM European call with maturity (the closed to) one year over time for the Dow Jones index



A similar graph can be found in Figure 6 for the three months to maturity ATM European call on the Dow Jones.

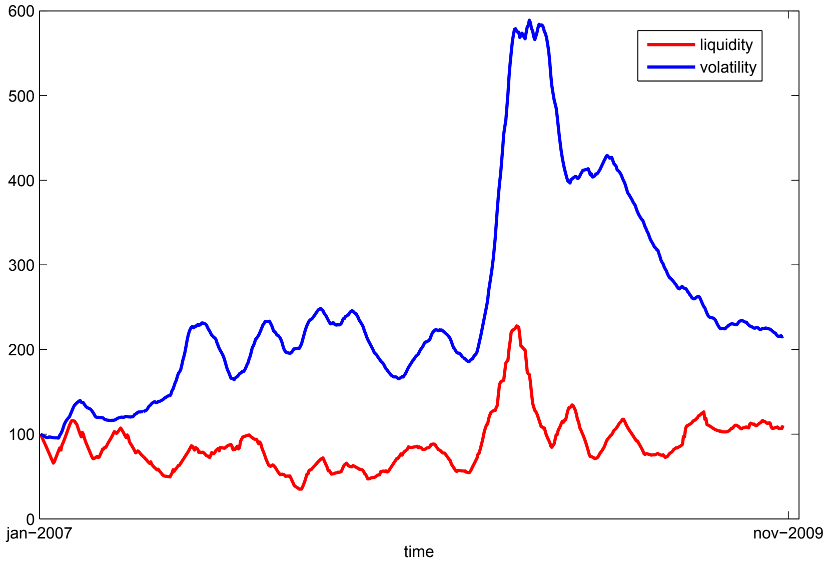
Figure 6 Implied liquidity for three months to maturity ATM European call on the Dow Jones



In Figure 7, one sees that (smoothed) VIX and implied liquidity both went up dramatically during the crisis, but that liquidity was reinstalled faster and volatility remained at a relatively higher level. VIX and implied liquidity are correlated measures related to market fear, but each of them shows a different fear ingredient (see also

Dhaene et al., 2012). Note that we have re-scaled numbers, such that the beginning of January 2007 is set to 100 and have taken a moving window average over VIX and implied liquidity over one month.

Figure 7 VIX (smoothed) and implied liquidity



3.2 *Liquidity sensitivity and hedging of liquidity risk*

Seen the utmost importance and the extreme dependence on liquidity of financial institutions, hedge funds and other financial players, claims contingent on liquidity may serve as potential hedges against drying up liquidity in critical times. Examples could be European calls on implied liquidity levels, useful for players facing redemption risk in periods of financial distress and which often comes along with less liquidity. Or Asian type or realised liquidity swaps, for financial players who are exposed to liquidity risk all the time, because of the periodically adjustment of hedges or certain dynamics trading strategies (e.g., CPPIs), could be interesting for macro hedge against liquidity over the time-period of interest. The variations are endless.

A new parameter, calls for also a new Greek. We call the sensitivity of the bid and ask price with respect to a change in the liquidity parameter λ, the *lidip*:

$$\text{lidip}_{bid} = \frac{\partial bid(X)}{\partial \lambda} \text{ and } \text{lidip}_{ask} = \frac{\partial ask(X)}{\partial \lambda}. \tag{5}$$

Lidip is negative for the bid and positive for the ask.

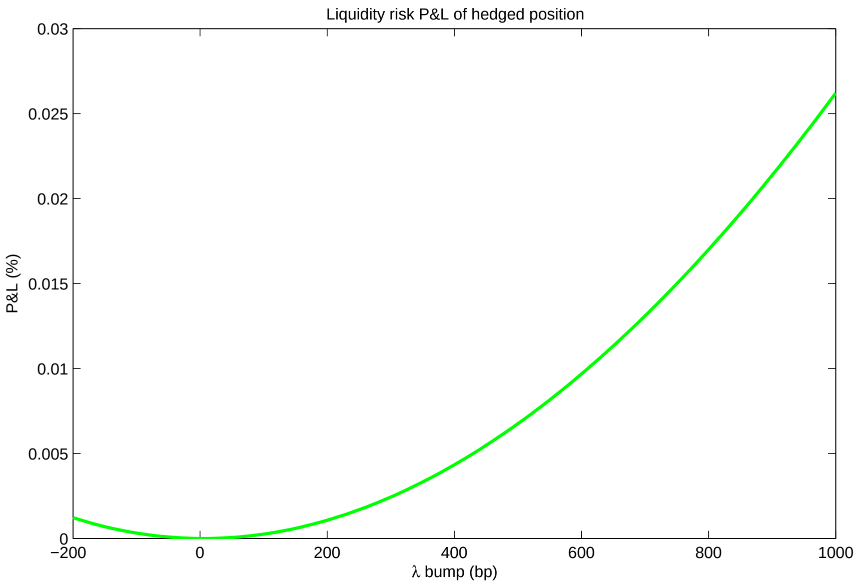
Next, we illustrate the use of this new Greek for the moment hypothetical liquidity derivative instruments. We first assume that a zero strike call on λ is traded and assume we are long an ATM European call. Pricing is for the illustration done under Black-Scholes. We have the following setting: $S_0 = 100$, $\sigma = 20\%$, $r = 0\%$, $q = 0\%$. We assume a certain current lambda, say 200 bp. Then bid, mid, ask price of the

European call are given by 7.5812, 7.9656 and 8.3633, respectively. Next, we bump lambda a bit, say by 10 bp, and recalculate the effect of this. By a finite difference approximation of equation (5), we calculate

$$\text{lidip}_{bid} = -18.7308 \text{ and } \text{lidip}_{ask} = 20.0849.$$

Since, we are long the call, we focus on the bid. In order to hedge for liquidity risk, we buy 18.7308 zero strike calls on lambda; these zero strike calls have notional 1 and these cost around 0.02. Imagine, that due to some instantaneous event, liquidity dried up, say lambda shot to 400 bp. Then leaving all other things unchanged, the bid, mid, ask price of the European call are given by 7.2151, 7.9656 and 8.7685, respectively. Since we book assets at bid and liabilities at ask, we hence note a loss of 7.5812 – 7.2151 = 0.3661 on the call, but our zero strike liquidity call is now priced 0.04 and we hold 18.7308 of them. We hence book a profit of 0.3746 (18.7308×(0.04 – 0.02)), which is almost equal to 0.3661 and which is actually a 0.11% profit with respect to the call mid price. A naked (no liquidity hedge) position would have given a loss of about 4.6%. In Figure 8, one can see similar P&Ls for different changes in lambda. In the left graph one sees a comparison of naked (red) and hedged (green) positions. Note that a profit in the naked position is booked if liquidity increases. The right figure is just a zoom in of the hedged P&L and very much looks like the P&L of a stock delta hedge position (gamma).

Figure 8 Hedging liquidity risk for ATM calls



We elaborate further on what it means to monetise the liquidity parameter. The situation is very similar to how we monetise volatility and make a book vega neutral. Vega tells us the value we expect to gain or lose if volatility were to move by a percentage point and we may wish to hedge this value by taking position with a particular vega exposure.

For liquidity one could do something similar. Assume a hypothetical option book that is short variance swaps on the liability side against which one holds an option portfolio accessing $-(2/T)\log(S_T/S_0)$. In the world of liquid markets with the law of one price and an underlying that diffuses, this is a perfect hedge and there are no losses of any kind. If we now switch to the world of two prices (bid and ask), it is important to realise that one should mark assets at bid prices and liabilities at ask prices. If all is marked to mid quote then a drop in liquidity lowers bids, raises asks and leaves mid quotes unchanged. There are then no financial consequences to liquidity drying up and no risks to hedge. This is wrong and we have witnessed the serious consequences to the financial industry and even broader than that of a dramatic change in liquidity during the credit crunch crisis.

Hence, in the world of two prices, one must mark the variance swap (and any other liability) at ask and mark the hedge portfolio at bid. In order to hedge perfectly, the difference should be held in capital reserves as a precaution against the necessity to unwind adversely the next day without any ability to hold to maturity.

Now, there is the risk of the liquidity drying up adversely raising the variance swap liability while it simultaneously lowers the value of the option portfolio. The difference to be held in capital reserves hence increases (due to effects on both sides: a lower bid and a higher ask). This is the money value of the liquidity exposure. Hedging liquidity hence boils down in looking for instruments that compensate for the extra capital needed. As in the example above, one can determine how many zero strike calls you have to hold or what your position should be in a contract that earns the bump in lambda, to do this. Short of trading and buying lambda directly through liquidity derivatives, what else can people do? An increase in lambda (liquidity drying up) cannot be hedged in any other direct way. This is not true for volatility exposure as we can hold the vega. Liquidity risk is in that sense special. Liquidity is a monetisable and real risk which without liquidity derivatives is unhedgeable.

4 Conclusions

In this paper we introduced the concept of implied (il)liquidity of vanilla options. Implied liquidity is based on the fundamental theory of conic finance, in which the one-price model is abandoned and replaced by a two-price model giving bid and ask prices for traded assets. The pricing is done by making use of distorted expectations. We worked with a one parameter distortion function representing the liquidity situation. After reviewing under the Black-Scholes setting the theory and numerics of the calculation of bid-ask prices under conic finance theory, we came to the concept of implied liquidity. In a fixed market with no movement in the cone of acceptable risks and hence no change in liquidity, the bid ask spread can move around non-linearly with maturity and/or volatility. Because the spread can move in a constant market with no change in liquidity, spread itself is not a perfect measure of liquidity. Implied liquidity can overcome this criticism. We have illustrated the theory on SP500 and Dow Jones index data and showed that for vanilla options we typically have for higher strikes (OTM) more implied illiquidity. We typically do not have much term structure. Also, we performed a historical study, in which we clearly see a serious drying up of liquidity in the weeks post the Lehman bankruptcy. Seen the evidence of changing liquidity in the recent past with a potentially very disruptive drying up of liquidity, liquidity derivative contracts could provide extra hedges for such circumstances. We observed that the above notion of implied liquidity leads toward a mean-reverting modelling of liquidity similar to stochastic volatility. We believe such stochastic liquidity modelling could be very useful in structured product pricing, delta-gamma-vega hedging studies and risk-management in general.

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