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Parametric Estimation on Constant Stress Partially Accelerated Life Tests for the Exponentiated Exponential Distribution using Multiple Censoring

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Abstract

If the items have high reliability then to check the lifetime of items under normal use condition takes more time and cost in comparison with the accelerated condition. The items put higher stress than the usual level of stresses to generate early failures in a short period to reduce the costs involved in the testing of items without any change in the quality. This study is based on constant stress partially accelerated life tests for Exponentiated Exponential distribution using multiple censoring schemes. The maximum likelihood estimates and asymptotic variance and covariance matrix are obtained. The confidence intervals for parameters are also constructed. At last, a simulation technique is used to check the performance of the estimators.

Keywords: Constant stress partially accelerated life tests, Exponentiated Exponential distribution, Multiple censoring, Fisher Information Matrix, a Simulation study.

I. Introduction

In the present market situation, the manufacturing designs are bettering day by day because there is a big change in technology. If an item has high reliability than it is too much tough to obtain information about the lifetime of items or products under normal usage condition at the time of testing. In this type of situation, the accelerated life test (ALT) is the best choice to get information on the life of the items or products. ALT is used to get information on items life or products life in a short period with a shortage of cost by testing them at accelerated conditions after this testing them on normal use conditions to induce early failures. These conditions are referred to as stresses. The stresses may be in the form of temperature, voltage, force, etc.

Normally, three types of stresses are applied in accelerated life testing, such as constant stress, step-stress, and progressive stress. Here we are focusing only on constant stress. In constant stress accelerated life test, the products or items are operated at fixed levels of stress throughout the testing. From ALT, two types of data are obtained, such as complete and censored data. In the complete data, the lifetime of each unit is known, but the lifetime of each unit is unknown in censored data. A mathematical model which is related to the lifetime of an item or product and stress is either known or can be assumed in ALT. There are many situations in which these relationships are unknown, and we can not conclude these relationships. This means that data can not be extrapolated to use conditions which are obtained from ALT. Such situations where the test items are run at both normal and higher than normal stress conditions, the partially accelerated life test (PALT) is used. In PALT, two main methods are used by reliability practitioners such as constant stress partially accelerated life test (CSPALT) and step-stress partially accelerated life test (SSPALT). The products or items are tested either usual or higher than usual condition until the test is ended in CSPALT.

In many situations, the lifetime experiment could out of control due to many reasons like components of a system may break accidentally. In type-I censoring (time censoring) scheme, the test is terminated after a fixed amount of time, and in type-II censoring (item censoring), the test is terminated after a fix proportion of items. As we know that the removal of items or components from the test during testing is possible in the progressive type censoring scheme, while type-I and type-II censoring schemes don't allow the removal of items or components from a test during testing. In this type of situation, the multiple censoring schemes are the best choice for an engineer or reliability practitioner because multiple censoring schemes allows the removal of items from the test during the testing at any situation or any time. We define multiple censoring schemes as when the testing of items or components fails because of more than one reason, then multiple censoring occurs. Tobias and Trindada [1] observed that the type-I and type-II censoring schemes are a special case of multiple censoring schemes.

There is much literature available on PALT with constant stress with many types of censoring schemes. *Abd El-Raheem et al. [2] presented a study on constant stress accelerated life test with the use of geometric process when the lifetime of test units follows Extention of Exponential distribution under the type-II progressive censoring scheme. Kamal et al. [3] presented a study on designing of partially accelerated life test when the lifetime of items follows Inverted Weibull distribution with constant stress under the type-I censoring scheme. Abdullah M. [4] dealt with parameters estimation when the lifetime of units follows Generalized Half Logistic distribution for progressive type-II censored data. Zhang and Fang [5] dealt with an estimation of acceleration factor when the lifetime of units follows Exponential distribution under CSPALT based on type-I censored data. A new approach of constructing the exact lower and upper confidence limits is proposed by them for the acceleration factor. Sadia and Islam [6] presented a study on CSPALT plans when the lifetime of units follows Rayleigh distribution based on type-II censored data. Tahani and Areej [7] dealt with an inference on CSPALT under progressive type-II censored data based on a mixture of Pareto distribution. Mohamed et al. [8] presented a study on CSPALT using progressive type-II censored data when the lifetime of items follows Modified Weibull distribution. They discussed two bootstrap confidence intervals, which are called bootstrap-t and bootstrap-p. Xiaolin and Yimin [9] presented a study on CSPALT using the masked series system when the lifetime of components follows Complementary Exponential distribution based on progressive type-II censoring. Ismail [10] presented a study on CSPALT for Weibull distribution based on hybrid censoring scheme. He makes a statistical inference by using two methods; maximum likelihood and percentile bootstrap method. Nassar and Elharoun [11] dealt with an inference on CSPALT for Exponentiated Weibull distribution in the case of multiple censored data. Amal et al. [12] presented a study on CSPALT for inverted Weibull distribution in the case of multiple censoring scheme. Cheng and Weng [13] estimated parameters under multiple censoring scheme when the lifetime of items follows Burr XII distribution.*

The paper organized as follows. The model description and test procedure are given in section II. The basic assumptions for CSPALT are also given in section II. The point Estimation is given in section III. In this section, the likelihood function of the model under multiple censoring schemes is observed, and the Fisher Information matrix is also investigated in this section. In section IV, the confidence intervals are developed. The simulation study is given in section V. Finally, the conclusions are made in section VI.

II. Model Description and Test Procedure

I. Exponentiated Exponential Model

The Exponentiated Exponential distribution is commonly known as the Generalized Exponential distribution. This distribution is a particular member of Exponentiated Weibull distribution under two parameters form [14]. It is quite effective in analyzing several lifetime data, mainly in place of Gamma and Weibull Distribution in two parameters case. The above three distributions coincide with Exponential distribution in one parameter form if the value of the shape parameter becomes one. The Exponentiated Exponential plays an important role in reliability analysis because of its simplicity. If the lifetime of the item follows the Exponentiated Exponential distribution, then the test procedure for CSPALT under multiple censoring schemes is as follows.

The probability density function (pdf) of Exponentiated Exponential distribution is given as

$$f_1(t_i) = \alpha\lambda e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1} \qquad t_i, \alpha, \lambda > 0 \ ; \ i = 1,2,...,n_1 \tag{1}$$

Where, α and λ are shape, scale parameters respectively. The ith observed lifetime of the test under normal condition item is denoted by t_i.

The cumulative density function (cdf) of Exponentiated Exponential distribution is given as

$$F_1(t_i) = (1-e^{-\lambda t_i})^{\alpha} \tag{2}$$

The reliability function of Exponentiated Exponential distribution is given as

$$R_1(t_i) = 1-(1-e^{-\lambda t_i})^{\alpha}$$

The hazard function of Exponentiated Exponential distribution is given as

$$H_1(t_i) = \frac{\alpha\lambda e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1}}{1-(1-e^{-\lambda t_i})^{\alpha}}$$

Under the accelerated condition, the probability density function (pdf) of a lifetime $X = \beta^{-1}T$ is given as

$$f_2(x_j) = \alpha\beta\lambda e^{-\lambda\beta x_{j_i}}(1-e^{-\lambda t\beta x_j})^{\alpha-1} \qquad t_i, \alpha, \lambda > 0, \ \beta > 1; \ j = 1,2,...,n_2 \tag{3}$$

Under the accelerated condition, the cumulative density function (cdf) of a lifetime $X = \beta^{-1}T$ is given as

$$F_2(x_j) = (1-e^{-\lambda t\beta x_j})^{\alpha} \tag{4}$$

The reliability function of a lifetime, $X = \beta^{-1}T$ under accelerated condition, is given as

$$R_2(x_j) = 1-(1-e^{-\lambda\beta x_j})^{\alpha}$$

The reliability function of a lifetime, $X = \beta^{-1}T$ under accelerated condition, is given as

$$H_2(x_j) = \frac{\alpha\beta\lambda e^{-\lambda\beta x_{j_i}}(1-e^{-\lambda t\beta x_j})^{\alpha-1}}{1-(1-e^{-\lambda\beta x_j})^{\alpha}}$$

Where x_j is jth observed lifetime under the case of the accelerated condition.

II. Assumptions

The basic assumptions for CSPALT are given as

- The lifetimes of items $T_i \;\; i = 1,2,...,n_1$ are independent and identically distributed random variable with probability density function given in equation (1), which is allocated to normal condition.
- The lifetimes of items $X_j \;\; j = 1,2,...,n_2$ are also independent and identically distributed random variable with probability density function given in equation (3), which is allocated to accelerated condition.
- T_i and X_i are mutually independent also.
- n_1 and n_2 are the total numbers of items at normal and accelerated condition, respectively.

III. Parameter Estimation

I. Point Estimates

In this section, we use the maximum likelihood (ML) technique for estimating parameters. ML technique is the most important technique for fitting the statistical model; it has many interesting properties like asymptotic unbiased, asymptotic efficiency and asymptotic normality, etc.

$t_{(1)} < t_{(2)} < ...t_{(n)}$ are supposed observed values of the total lifetime T at the normal condition and $t_{(1)} < t_{(2)} < ...t_{(n)}$ are the supposed observed values of the lifetime X at the accelerated condition.

Then the likelihood of Exponentiated Exponential distribution under multiple censored data is given as

$$L(t_i,\alpha,\lambda,\beta) = \prod_{i=1}^{n}\left[f_1(t_i)\right]^{\delta_i,1,f}\left[1-F_i(t_i)\right]^{\delta_i,1,c} \times \left[f_2(x_i)\right]^{\delta_i,2,f}\left[1-F_2(x_i)\right]^{\delta_i,2,c} \tag{5}$$

$$L(t_i,\alpha,\lambda,\beta) = \prod_{i=1}^{n}\left[\alpha\lambda e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1}\right]^{\delta_i,1,f}\left[1-(1-e^{-\lambda t_i})^{\alpha}\right]^{\delta_i,1.c}\left[\alpha\beta\lambda e^{-\lambda\beta x_{j_i}}(1-e^{-\lambda t\beta x_j})^{\alpha-1}\right]^{\delta_i,2,f}\left[1-(1-e^{-\lambda t\beta x_j})^{\alpha}\right]^{\delta_i,2.c}$$

$\delta_{i,1,f}$, $\delta_{i,1,c}$, $\delta_{i,2,f}$, $\delta_{i,2,c}$ are indicator functions. The values of indicator functions are given as

$$\delta_{i,1,f},\delta_{i,2,f} = \begin{cases} 1 \; the item \; failed at \; stress condition \\ 0 \;\; otherwise \end{cases}$$

$$\delta_{i,1,c},\delta_{i,2,c} = \begin{cases} 1 \; the item censored at \; normal condition \\ 0 \;\; otherwise \end{cases}$$

$$\sum_{i=1}^{n}\delta_{i,1,f} = n_{1f} = Number of \;\; failed items at \; nomal condition$$

$$\sum_{i=1}^{n}\delta_{i,2,f} = n_{2f} = Number of \;\; failed items at \; accelerated \; condition$$

$$\sum_{i=1}^{n}\delta_{i,1,c} = n_{1c} = Number of\ censored item at\ normal condition$$

$$\sum_{i=1}^{n}\delta_{i,2,c} = n_{2c} = Number of\ censored item at\ accelerated\ condition$$

$$n_f = n_{1f} + n_{2f}$$

The log-likelihood function is simply the natural logarithm of the likelihood function and given as

$$\begin{aligned}\ln L = &\sum_{i=1}^{n}\delta_{i,1,f}\left[\ln\alpha + \ln\lambda - \lambda t_i + (\alpha-1)\ln(1-e^{-\lambda t_i})\right] + \sum_{i=1}^{n}\delta_{i,1,c}\ln\left[1-(1-e^{-\lambda t_i})^{\alpha}\right] \\ &+ \sum_{i=1}^{n}\delta_{i,2,f}\left[\ln\alpha + \ln\beta + \ln\lambda - \lambda\beta x_i + (\alpha-1)\ln(1-e^{-\lambda\beta x_i})\right] + \sum_{i=1}^{n}\delta_{i,2,c}\ln\left[1-(1-e^{-\lambda x_i})^{\alpha}\right]\end{aligned} \tag{6}$$

Where $L(t_i, \alpha, \lambda, \beta) = \ln L$

The MLEs of α, λ and β are obtained by differentiating log-likelihood function concerning α, λ and β respectively and equating to zero. Then the equations are given as

$$\begin{aligned}\frac{\partial \ln L}{\partial \alpha} = &\left[\frac{n_{1f}}{\alpha} + \sum_{i=1}^{n}\delta_{i,1,f}\ln(1-e^{-\lambda t_i})\right] - \sum_{i=1}^{n}\delta_{i,1,c}\frac{(1-e^{-\lambda t_i})^{\alpha}\ln(1-e^{-\lambda t_i})}{\left[1-(1-e^{-\lambda t_i})^{\alpha}\right]} \\ &+ \left[\frac{n_{2f}}{\alpha} + \sum_{i=1}^{n}\delta_{i,2,f}\ln(1-e^{-\lambda\beta x_i})\right] - \sum_{i=1}^{n}\delta_{i,2,c}\left[\frac{(1-e^{-\lambda\beta x_i})\ln(1-e^{-\lambda\beta x_i})}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right]\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \lambda} = &\left[\frac{n_{1f}}{\lambda} - \sum_{i=1}^{n}\delta_{i,1,f}t_i + (\alpha-1)\sum_{i=1}^{n}\delta_{i,1,f}\frac{e^{-\lambda t_i}t_i}{1-e^{-\lambda t_i}}\right] + \sum_{i=1}^{n}\delta_{i,1,c}\frac{\alpha(1-e^{-\lambda t_i})^{\alpha}e^{-\lambda t_i}t_i}{\left[1-(1-e^{-\lambda t_i})^{\alpha}\right]} \\ &+ \left[\frac{n_{2f}}{\lambda} - \beta\sum_{i=1}^{n}\delta_{i,2,f}x_i + (\alpha-1)\sum_{i=1}^{n}\delta_{i,2,f}\frac{e^{-\lambda\beta x_i}\beta x_i}{1-e^{-\lambda\beta x_i}}\right] - \sum_{i=1}^{n}\delta_{i,2,c}\left[\frac{\alpha(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}\beta x_i}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right]\end{aligned} \tag{8}$$

$$\frac{\partial \ln L}{\partial \beta} = \left[\frac{n_{1f}}{\beta} - \lambda\sum_{i=1}^{n}\delta_{i,2,f}x_i + (\alpha-1)\sum_{i=1}^{n}\delta_{i,2,f}\frac{e^{-\lambda\beta x_i}\lambda x_i}{1-e^{-\lambda\beta x_i}}\right] - \sum_{i=1}^{n}\delta_{i,2,c}\left[\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\lambda x_i}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right] \tag{9}$$

There is no closed solution of these nonlinear equations. So we use the Newton-Raphson technique for solving these equations.

II. Fisher Information Matrix

The Fisher Information matrix is the composition of negative second partial derivatives of log-likelihood function and can be expressed as

$$I = \begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial \alpha^2} & -\dfrac{\partial^2 \ln L}{\partial \alpha \partial \lambda} & -\dfrac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\dfrac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\dfrac{\partial^2 \ln L}{\partial \lambda^2} & -\dfrac{\partial^2 \ln L}{\partial \lambda \partial \beta} \\ -\dfrac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\dfrac{\partial^2 \ln L}{\partial \beta \partial \lambda} & -\dfrac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix} \tag{10}$$

The elements of Fisher-Information matrix is given as

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \sum_{i=1}^{n} \delta_{i,1,f}\left[-\frac{1}{\alpha^2}\right] - \sum_{i=1}^{n} \delta_{i,1,c} \frac{(1-e^{-\lambda t_i})^{\alpha} \ln(1-e^{-\lambda t_i})}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\left[\ln(1-e^{-\lambda t_i}) + \frac{(1-e^{-\lambda t_i})^{\alpha} \ln(1-e^{-\lambda t_i})}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\right]$$
$$+\sum_{i=1}^{n} \delta_{i,2,f}\left[-\frac{1}{\alpha^2}\right] - \sum_{i=1}^{n} \delta_{i,2,c} \frac{\ln(1-e^{-\lambda\beta x_i})}{\left\{1-(1-e^{-\lambda\beta x_i})\right\}}\left[\ln(1-e^{-\lambda\beta x_i}) + \frac{(1-e^{-\lambda\beta x_i}) \ln(1-e^{-\lambda\beta x_i})}{\left[1-(1-e^{-\lambda\beta x_i})^{\alpha}\right]}\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \delta_{i,1,f}\left[-\frac{e^{-\lambda t_i} t_i}{1-e^{-\lambda t_i}}\right]$$
$$+\sum_{i=1}^{n} \delta_{i,1,c}\left[\frac{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}\left\{(1-e^{-\lambda t_i})^{\alpha} e^{-\lambda t_i} t_i + (1-e^{-\lambda t_i})^{\alpha-1}\ln(1-e^{-\lambda t_i}) e^{-\lambda t_i} t_i\right\} + \left\{t_i (1-e^{-\lambda t_i})^{2\alpha-1} \ln(1-e^{-\lambda t_i}) e^{-\lambda t_i}\right\}}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}^2}\right]$$
$$-\sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})}\right] + \sum_{i=1}^{n} \delta_{i,2,c}\left[\frac{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}\left\{(1-e^{-\lambda\beta x_i})^{\alpha-1} e^{-\lambda\beta x_i} + (1-e^{-\lambda\beta x_i})^{\alpha-1}\ln(1-e^{-\lambda\beta x_i}) e^{-\lambda\beta x_i} \beta_i\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right.$$
$$\left.+\frac{\left\{\beta x_i (1-e^{-\lambda\beta x_i})^{2\alpha-1} \ln(1-e^{-\lambda\beta x_i}) e^{-\lambda\beta x_i}\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i} \lambda x_i}{(1-e^{-\lambda\beta x_i})}\right] - \sum_{i=1}^{n} \delta_{i,2,c}\left[\frac{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha-1} e^{-\lambda\beta x_i} - (1-e^{-\lambda\beta x_i})^{\alpha-1}\lambda x_i\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right.$$
$$\left.+\frac{\left\{(1-e^{-\lambda\beta x_i})^{2\alpha-1} \ln(1-e^{-\lambda\beta x_i}) \lambda x_i e^{-\lambda\beta x_i}\right\}}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}^2}\right]$$

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = \sum_{i=1}^{n} \delta_{i,1,f}\left[-\frac{e^{-\lambda t_i} t_i}{1-e^{-\lambda t_i}}\right] + \sum_{i=1}^{n} \delta_{i,1,c}\left[\frac{t_i e^{-\lambda t_i}(1-e^{-\lambda t_i})^{\alpha-1}}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\left\{\alpha^{-1} + \ln(1-e^{-\lambda t_i}) + \frac{(1-e^{-\lambda t_i})^{\alpha}\ln(1-e^{-\lambda t_i})}{\left\{1-(1-e^{-\lambda t_i})^{\alpha}\right\}}\right\}\right]$$
$$+\sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}\beta x_i}{(1-e^{-\lambda\beta x_i})}\right] + \sum_{i=1}^{n} \delta_{i,2,c}\left[\frac{\beta x_i e^{-\lambda\beta x_i}(1-e^{-\lambda\beta x_i})}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\left\{\alpha^{-1} + \ln(1-e^{-\lambda t\beta x_i}) + \frac{(1-e^{-\lambda\beta x_i})^{\alpha}\ln(1-e^{-\lambda t\beta x_i})}{\left\{1-(1-e^{-\lambda\beta x_i})^{\alpha}\right\}}\right\}\right]$$
$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\sum_{i=1}^{n} \delta_{i,1,f}\left[\frac{1}{\lambda^2} + (\alpha-1)\frac{e^{-\lambda t_i} t_i}{1-e^{-\lambda t_i}}\left\{\frac{t_i + e^{-\lambda t_i}}{1-e^{-\lambda t_i}}\right\}\right]$$
$$+\sum_{i=1}^{n} \delta_{i,1,c}\alpha t_i\left[\frac{(1-e^{-\lambda t_i})e^{-\lambda t_i}}{1-(1-e^{-\lambda t_i})^{\alpha}}\left\{\frac{(\alpha-1)(1-e^{-\lambda t_i})^{\alpha-2} t_i e^{-\lambda t_i}}{(1-e^{-\lambda t_i})^{\alpha-1}} - t_i + \frac{\alpha e^{-\lambda t_i}(1-e^{-\lambda t_i})}{1-(1-e^{-\lambda t_i})^{\alpha}}\right\}\right]$$
$$-\sum_{i=1}^{n} \delta_{i,2,f}\left[\frac{1}{\lambda^2} + (\alpha-1)\frac{e^{-\lambda\beta x_i} t_i}{1-e^{-\lambda\beta x_i}}\left\{\frac{\beta x_i + e^{-\lambda\beta x_i}}{1-e^{-\lambda\beta x_i}}\right\}\right]$$
$$+\sum_{i=1}^{n} \delta_{i,2,c}\alpha\beta x_i\left[\frac{(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\left\{\frac{(\alpha-1)(1-e^{-\lambda\beta x_i})^{\alpha-2} - \lambda\beta x_i e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})^{\alpha-1}} - \lambda\beta x_i + \frac{\alpha e^{-\lambda\beta x_i}(1-e^{-\lambda\beta x_i})}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\right\}\right]$$
$$\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} = \sum_{i=1}^{n} \delta_{i,2,f}\left[-x_i + (\alpha-1)\frac{e^{-\lambda\beta x_i}\beta x_i}{1-e^{-\lambda\beta x_i}}\left\{-\lambda x_i + \frac{1}{\beta} - \frac{\lambda x_i e^{-\lambda\beta x_i}}{1-e^{-\lambda\beta x_i}}\right\}\right]$$
$$+\sum_{i=1}^{n} \delta_{i,2,c}\frac{\alpha(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}\beta x_i}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\left[\frac{e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})} - \lambda x_i + \frac{1}{\beta} + \frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\lambda x_i}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\alpha}=\sum_{i=1}^{n}\delta_{i,2,f}\left[\frac{e^{-\lambda\beta x_i}\lambda x_i}{1-e^{-\lambda\beta x_i}}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,c}e^{-\lambda\beta x_i}\lambda x_i\left[\left\{\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}}{1-(1-e^{-\lambda\beta x_i})}\right\}\left\{\frac{1}{\alpha}+\ln(1-e^{-\lambda\beta x_i})+\frac{(1-e^{-\lambda\beta x_i})^{\alpha}\ln(1-e^{-\lambda\beta x_i})}{1-(1-e^{-\lambda\beta x_i})^{\alpha}}\right\}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\lambda}=\sum_{i=1}^{n}\delta_{i,2,f}\left[-x_i+(\alpha-1)x_i\frac{\lambda x_i}{1-e^{-\lambda\beta x_i}}\left\{\frac{1}{\lambda}-\beta x_i-\frac{e^{-\lambda\beta x_i}\beta x_i}{1-e^{-\lambda\beta x_i}}\right\}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,c}\alpha x_i\left[\left\{\frac{\lambda(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}}{1-(1-e^{-\lambda\beta x_i})}\right\}\left\{\frac{1}{\lambda}-\beta x_i+\frac{(\alpha-1)(1-e^{-\lambda\beta x_i})^{\alpha-2}e^{-\lambda\beta x_i}\beta x_i}{(1-e^{-\lambda\beta x_i})^{\alpha-1}}+\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\beta x_i}{1-(1-e^{-\lambda\beta x_i})^{\alpha-1}}\right\}\right]$$

$$\frac{\partial^2 \ln L}{\partial\beta^2}=\sum_{i=1}^{n}\delta_{i,2,f}\left[-\frac{1}{\beta^2}+(\alpha-1)\lambda x_i\frac{e^{-\lambda\beta x_i}}{1-e^{-\lambda\beta x_i}}\left\{-\lambda x_i-\frac{e^{-\lambda\beta x_i}\lambda x_i}{1-e^{-\lambda\beta x_i}}\right\}\right]$$
$$+\sum_{i=1}^{n}\delta_{i,2,c}\alpha\lambda x_i\left[\left\{\frac{(1-e^{-\lambda\beta x_i})e^{-\lambda\beta x_i}}{1-(1-e^{-\lambda\beta x_i})^{\alpha-1}}\right\}\left\{\frac{\lambda x_i e^{-\lambda\beta x_i}}{(1-e^{-\lambda\beta x_i})}-\lambda x_i+\frac{\alpha(1-e^{-\lambda\beta x_i})^{\alpha-1}e^{-\lambda\beta x_i}\lambda x_i}{(1-e^{-\lambda\beta x_i})^{\alpha}}\right\}\right]$$

The asymptotic variance-covariance is simply obtained by taking the inverse of the Fisher Information matrix. The asymptotic variance-covariance is given as

$$\Sigma=I^{-1}=\begin{bmatrix}-\frac{\partial^2\ln L}{\partial\alpha^2} & -\frac{\partial^2\ln L}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ln L}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ln L}{\partial\lambda^2} & -\frac{\partial^2\ln L}{\partial\lambda\partial\beta}\\ -\frac{\partial^2\ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2\ln L}{\partial\beta\partial\lambda\partial} & -\frac{\partial^2\ln L}{\partial\beta^2}\end{bmatrix}^{-1}=\begin{bmatrix}AVar(\hat\alpha) & ACov(\hat\alpha\hat\lambda) & ACov(\hat\alpha\hat\beta)\\ ACov(\hat\lambda\hat\alpha) & AVar(\hat\lambda) & ACov(\hat\lambda\hat\beta)\\ ACov(\hat\beta\hat\alpha) & ACov(\hat\beta\hat\lambda) & AVar(\hat\beta)\end{bmatrix} \tag{11}$$

Where, $AVar$ and $ACov$ stand for asymptotic variance, asymptotic covariance respectively.

IV. Interval Estimates

A confidence interval for parameters is a type of interval estimate, computed from the statistics of the observed data, that consist of the accurate value of an unknown population parameter. In other words, a confidence interval is simply the probability. So, a confidence interval means the probability that the value of a parameter will fall between the lower and upper bound of a probability distribution. Mostly, 90%, 95%, and 99% confidence levels are used.

The two-sided confidence limits can be constructed as

$$p\left[-z\le\frac{\hat\varphi-\varphi}{\sigma(\hat\varphi)}\le z\right]=1-\kappa \tag{13}$$

This construction of two-sided confidence limits is for the maximum likelihood estimate $\hat\varphi$ of a population parameter $\psi=(\alpha,\lambda,\beta)$. In the above equation (13), z stands for $100(1-\kappa/2)$ the standard normal percentile and κ stands for the significance level. So, for a population parameter

φ, an appropriate confidence limits can be obtained, such that

$$p[\hat{\varphi} - z\sigma(\hat{\varphi}) \le \varphi \le \hat{\varphi} + z\sigma(\hat{\varphi})] = 1 - \kappa$$

Where, lower confidence limit $L_{\varphi} = \hat{\varphi} - z\sigma(\hat{\varphi})$ and upper confidence limit $U_{\varphi} = \hat{\varphi} + z\sigma(\hat{\varphi})$

V. Simulation Study

In this section, we perform a simulation study to check the performance of the estimators having Exponentiated Exponential distribution using multiple censored data. This simulation study is done Monte Carlo Simulation technique by using R-Software. The means square error and bias are estimated to check the performance of estimators. The following steps are made for this simulation study.

- First, we divide the total sample n into two parts, n_1 and n_2. $n_1 = n\pi$ and $n_2 = n(1-\pi)$
- Generate $t_{1,1} < t_{2,2} < ... < t_{n_1,1}$ and $t_{2,1} < t_{2,2} < ... < t_{n_2,2}$ random samples of size n_1 and n_2 in normal and stress condition respectively from Exponentiated Exponential distribution.
- We generate 1000 random of size 50, 100, 150 and 200 and choose the values of the parameters as Case (I) $(\alpha = 0.6, \lambda = 0.6, \beta = 1.6)$, Case (II) $(\alpha = 0.6, \lambda = 0.6, \beta = 1.8)$ Case (III) $(\alpha = 0.4, \lambda = 0.8, \beta = 1.6)$, Case (IV) $(\alpha = 0.4, \lambda = 0.8, \beta = 1.8)$
- The acceleration factor and the distribution parameters are obtained for each sample and each set of parameters. The asymptotic variance and covariance matrix are also obtained for each set of parameters.
- Finally, for confidence levels $\gamma = 95\%, 99\%$ of acceleration factor, the two sides confidence limits and two parameters are constructed with the use of equation (13) for parameters α, λ and β.

Table 1: The values of Bias and MSE under the different size of samples for multiple censored data

n	Parameters	Case I $(\alpha = 0.6, \lambda = 0.6, \beta = 1.6)$			Case II $(\alpha = 0.6, \lambda = 0.6, \beta = 1.8)$		
		Estimates	Bias	MSE	Estimates	Bias	MSE
50	α	0.638	0.321	0.082	0.712	0.302	0.098
	λ	0.812	0.083	0.023	0.912	0.098	0.036
	β	1.321	0.068	0.235	1.543	0.076	0.243
100	α	0.616	0.310	0.092	0.743	0.298	0.094
	λ	0.823	0.078	0.019	0.843	0.088	0.034
	β	1.313	0.576	0.206	1.654	0.0702	0.224
150	α	0.602	0.297	0.076	0.765	0.287	0.087
	λ	0.801	0.065	0.014	0.921	0.784	0.045
	β	1.304	0.521	0.184	1.432	0.687	0.286
200	α	0.602	0.288	0.071	0.700	0.301	0.900
	λ	0.792	0.075	0.011	0.933	0.654	0.028
	β	1.297	0.543	0.098	1.876	0.765	0.198

Table 2: The values of Bias and MSE under the different size of samples for multiple censored data

n	Parameters	Case III $(\alpha = 0.4, \lambda = 0.8, \beta = 1.6)$			Case IV $(\alpha = 0.4, \lambda = 0.8, \beta = 1.8)$		
		Estimates	Bias	MSE	Estimates	Bias	MSE
50	α	0.543	0.289	0.064	0.612	0.598	0.078
	λ	0.865	0.265	0.054	0.923	0.336	0.067
	β	1.323	0.086	0.342	1.257	0.089	0.476
100	α	0.564	0.265	0.608	0.645	0.566	0.065
	λ	0.843	0.200	0.046	0.946	0.289	0.065
	β	1.456	0.076	0.298	1.345	0.081	0.398
150	α	0.486	0.286	0.586	0.596	0.500	0.054
	λ	0.802	0.202	0.065	0.897	0.288	0.058
	β	1.487	0.065	0.299	1.446	0.076	0.411
200	α	0.598	0.254	0.566	0.665	0.456	0.066
	λ	0.843	0.198	0.0421	0.886	0.328	0.048
	β	1.543	0.076	0.256	1.225	0.067	0.356

Table 3: Asymptotic Variance and Covariance Matrix of Estimators for Different Size of Samples under Multiple Censored Data

n	Parameters	Case I $(\alpha = 0.6, \lambda = 0.6, \beta = 1.6)$			Case II $(\alpha = 0.6, \lambda = 0.6, \beta = 1.8)$		
		α	λ	β	α	λ	β
50	α	0.00632	0.00226	0.00456	0.00776	0.00211	0.00509
	λ	0.00321	-0.00784	0.00387	0.00224	0.00449	0.00277
	β	0.00437	0.00298	0.04541	0.00443	0.00109	0.00118
100	α	0.00576	0.00276	0.00432	0.00654	0.00210	0.00498
	λ	0.00227	-0.00876	0.00267	0.00221	0.00265	0.00176
	β	0.00338	0.00234	0.00453	0.00343	0.00025	0.00101
150	α	0.00465	0.00199	0.00365	0.00554	0.00189	0.00334
	λ	0.00176	-0.00998	0.00223	0.00176	0.00228	0.00116
	β	0.00225	0.00178	0.00116	0.00225	-0.00987	-0.00554
200	α	0.00356	0.00113	0.00294	0.00445	0.00156	0.00223
	λ	0.00114	-0.00887	0.00132	0.00114	0.00189	0.00115
	β	0.00115	0.00117	0.00101	0.00112	-0.00998	-0.00776

Table 4: Asymptotic Variance and Covariance Matrix of Estimators for Different Size of Samples under Multiple Censored Data

n	Parameters	Case III $(\alpha=0.4,\lambda=0.8,\beta=1.6)$			Case IV $(\alpha=0.4,\lambda=0.8,\beta=1.8)$		
		α	λ	β	α	λ	β
50	α	0.00332	0.00098	0.00543	0.00376	0.00076	0.00432
	λ	0.00254	0.00221	0.00065	0.00577	0.00981	0.00087
	β	0.00443	0.00545	0.00334	-0.00654	0.00443	-0.00043
100	α	0.00224	0.00065	0.00332	0.00331	0.00054	0.00224
	λ	0.00223	0.00188	0.00045	0.00443	0.00076	0.00066
	β	0.00376	0.00332	0.00224	-0.00765	0.00411	-0.00066
150	α	0.00202	0.00043	0.00223	0.00269	0.00044	0.00187
	λ	0.00123	0.00117	0.00032	0.00332	0.00387	0.00054
	β	0.00321	0.00212	-0.00987	-0.00799	0.00332	-0.00098
200	α	0.00187	0.00011	0.00165	0.00211	0.00012	0.00112
	λ	0.00115	0.00076	0.00011	0.00287	0.00225	0.00043
	β	0.00234	0.00133	-0.00999	-0.00998	0.00225	-0.00076

Table 5: At Confidence Level $\kappa = 95\%, 99\%$, the Confidence Bounds of Estimates at Different Size of Samples

n	Parameters	Case I $(\alpha=0.4,\lambda=0.8,\beta=1.6)$				σ	Case I $(\alpha=0.4,\lambda=0.8,\beta=1.8)$				σ
		Confidence Interval $z=1.96$		Confidence Interval $z=2.58$			Confidence Interval $z=1.96$		Confidence Interval $z=2.58$		
		Lower Bound	Upper Bound	Lower Bound	Upper Bound		Lower Bound	Upper Bound	Lower Bound	Upper Bound	
50	α	0.57	0.73	0.53	0.89	0.08	0.51	0.78	0.61	0.93	0.07
	λ	0.68	0.89	0.57	0.76	0.04	0.55	0.86	0.67	0.83	0.10
	β	0.88	1.32	0.66	0.91	0.38	0.79	1.89	0.87	1.90	0.32
100	α	0.59	0.67	0.55	0.84	0.09	0.57	0.84	0.73	0.99	0.09
	λ	0.61	0.75	0.66	0.80	0.06	0.61	0.82	0.62	0.79	0.06
	β	0.77	1.34	0.73	0.88	0.43	0.98	1.56	0.97	2.11	0.35
150	α	0.64	0.71	0.67	0.81	0.06	0.44	0.60	0.65	0.76	0.05
	λ	0.64	0.76	0.58	0.72	0.09	0.87	0.93	0.56	0.69	0.08
	β	0.79	1.22	0.69	0.81	0.48	0.78	1.23	0.67	1.36	0.42
200	α	0.59	0.67	0.71	0.79	0.08	0.56	0.65	0.54	0.67	0.06
	λ	0.55	0.63	0.69	0.82	0.03	0.74	0.82	0.65	0.76	0.09
	β	0.88	1.01	0.61	0.73	0.35	0.77	1.02	0.68	1.11	0.36

Table 6: At Confidence Level $\kappa = 95\%, 99\%$, the Confidence Bounds of Estimates at Different Size of Samples

n	Parameters	Case I ($\alpha=0.4, \lambda=0.8, \beta=1.6$) Confidence Interval $z=1.96$ Lower Bound	Upper Bound	Confidence Interval $z=2.58$ Lower Bound	Upper Bound	σ	Case I V ($\alpha=0.4, \lambda=0.8, \beta=1.8$) Confidence Interval $z=1.96$ Lower Bound	Upper Bound	Confidence Interval $z=2.58$ Lower Bound	Upper Bound	σ
50	α	0.61	0.78	0.56	0.94	0.04	0.57	0.78	0.58	0.84	0.09
	λ	0.65	0.86	0.61	0.79	0.06	0.55	0.78	0.69	0.95	0.12
	β	0.78	1.22	0.61	0.85	0.32	0.72	1.86	0.75	1.60	0.39
100	α	0.55	0.69	0.61	0.79	0.08	0.68	0.94	0.64	0.79	0.08
	λ	0.56	0.71	0.77	0.89	0.07	0.68	0.87	0.62	0.79	0.10
	β	0.66	1.23	0.63	0.78	0.37	0.78	1.36	0.97	2.11	0.42
150	α	0.61	0.72	0.63	0.71	0.05	0.55	0.68	0.69	0.81	0.03
	λ	0.54	0.65	0.61	0.69	0.08	0.78	0.85	0.56	0.69	0.13
	β	0.68	1.10	0.79	0.85	0.42	0.67	1.11	0.67	1.36	0.47
200	α	0.55	0.62	0.65	0.72	0.02	0.64	0.71	0.65	0.69	0.07
	λ	0.59	0.66	0.65	0.76	0.05	0.64	0.69	0.68	0.79	0.02
	β	0.72	0.81	0.69	0.76	0.37	0.86	1.02	0.79	1.19	0.39

VI. Conclusions

This paper presented an inference on constant stress partially accelerated life tests for the Exponentiated Exponential distribution using multiple censoring schemes. The following observations are made based on the simulation study. The observations are

- In the table (1) and (2), the MSE and bias of estimators are obtained in four cases, and we can observe that the sample size increases the values of bias and MSEs decreases. The maximum likelihood estimates have good statistical properties for all sets of parameters because this set has the smallest biases for all sample sizes.
- In the table (3) and (4), the asymptotic variance and covariance matrix are obtained, and we can observe that the asymptotic variance-covariance of estimators decreases as sample size increases for the all sets of parameters.
- In the table (5) and (6), the confidence limits of the intervals for the parameters and the acceleration factor at 95% and 99% are obtained. The standard deviation (σ) of estimators is also obtained. We can observe that the width of the interval decreases as sample size increases for all sets of parameters.

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