An Optimal Ordering Policy for Non-instantaneous Deteriorating Items with Conditionally Permissible Delay in Payment under Two Storage Management

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ABSTRACT
In the present market scenario, trade credit financing has drawn much attention of various researchers. To increase sales, supplier/wholesalers offers some interest free period to their retailers. According to such consideration, in this paper a two warehouse inventory model for non-instantaneous deteriorating items with combination of different deterioration rate is developed. Shortages are not permissible. This paper mainly deals with non-instantaneous deteriorating items and trade credit financing with objective to derive the optimal replenishment policy that minimizes the average relevant inventory cost of the retailer. This model deals with single item only. Numerical examples are presented to validate the model. Sensitivity analysis has been performed by changing value of a parameter at a time and keeping value of rest parameters unchanged to study the effect on the inventory model.

Keywords
Two warehouses, Non-instantaneous deterioration, Permissible delay in payment.

1. INTRODUCTION
In the past, researchers have established a lot of research in the field of Inventory management and Inventory control system. Inventory management and control system basically deals with demand and supply chain problems and for this, production units (Producer of finished goods), venders, suppliers and retailers need to store the raw materials, finished goods for future demand and supply in the market. Many models have been developed considering various time dependent demand with shortages and without shortage. Hartely[1] discussed an inventory model with two storage facilities**.** Ghare and Schrader[2] initially worked in this field and they extended Harris[3], EOQ model with deterioration and shortages**.**

In many literatures, deterioration phenomenon has taken into account. Since many items are deteriorate with time, some instantly and some after a fixed life time of its own. Assuming the deterioration in both warehouses**,** Sarma[4], extended his earlier model to the case of infinite replenishment rate with shortages. Bhunia and Maiti[5] extended the model of Goswami and Chaudhary[6], in that model they were not consider the deterioration but shortages were allowed and backlogged. Many of researchers have studied with instantaneous deteriorating items.There are some items which do not deteriorate instantly and such items termed as “Non-instantaneous” deteriorating items. In real life mostly goods have a span of time maintaining original condition and during that time there is no deterioration occurs. In some real situations, there are many commodities such as wooden furniture, steel furniture, and fridge electric and electronic goods etc. which are not deteriorated instantly and may damage, spoiled due to bad handling and expiry of self-life period. K.S.Wu et al.[7] defined a new phenomenon as non-instantaneous deteriorating and considered the problem of determining the optimal replenishment policy for such items with stock dependent demand. Soon,L.Y.Ouyang et.al.[8] further developed an inventory model for non-instantaneous deteriorating items with permissible delay in payment. In the literature it is assumed that the deterioration rate in the both warehouse are of same type i.e. either constant or time dependent but it is not always true. It may vary in both warehouses depending upon the facilities provided there at. In the present market, rate of production of goods are very high due to the advance technologies and it results a very cut-throat competitive market and therefore, no company wants to reduce its sales as a large number of alternative products are available with additional features. Thus, to increase the sales, supplier offers a period to delay the payment, basically known as “Trade credit financing”. During permissible delay period offered by the supplier to the retailers, the retailer has not to pay any interest charges and after the end of the permissible delay period, he has to pay some interest, charges on the amount financed. During the permissible delay period retailer accumulate money by earning interest on sales revenue to reduce his total inventory cost and therefore, he needs more products and purchases it in bulk. Another case, of inadequate storage area, can occur when a procurement of a large amount of items is decided. That could be due to, an attractive price discount for bulk purchase which is available or, when the cost of procuring goods is higher than the other inventory related costs or, when demand for items is very high or, when the item under consideration is a seasonal product such as the yield of a harvest or, when there are some problems in frequent procurement. Since the retailer has limited storage space therefore he needs more spaces to store the product purchased during permissible period and hence required another storage house. In the busy market places due to the non-availability of space, retailer may rent a warehouse away from his retail shop for a short period. The trade credit financing problem was first discussed by Haley and Higgins[9].Then Goyel[10] developed an economic order quantity (EOQ) model under the condition of permissible delay in payments.Aggrawal and Jaggi[11] extended the Goel’s model. Jamal et. al.[12] further generalised the model by allowing the completely backlogged shortages. Thereafter much work has been done by several researchers. In this connection, the work of Hwang and Shin[13],Chang et al[14],Abad and Jaggi[15],Oyeang et al.[16],Huang[17],Liao[18], Jaggi and Khanna[19],Jaggi and Kausar[20], Jaggi and Mittal[21] and others are worth mentioning. However they have developed the model for a single ware-house under the assumption that the available ware-house has unlimited capacity. This assumption is not realistic as a ware-house is of limited capacity. As mentioned above at various situation retailers needs extra storage space to store the goods.

In this paper a deterministic inventory model for non-instantaneous deteriorating items with two level of storage and constant demand is developed under consideration that delay in payment is permitted. Further it is assumed that items are deteriorated after a fixed time period and deterioration rate in the both warehouses are different and deterioration cost is taken equal in both warehouses. Stock is transferred from RW to OW under continuous release pattern and the transportation cost is incurred. Different cases, depending upon the permissible delay period offered by supplier are discussed and results are compared with the help of numerical examples. The remaining paper is organized as follows: In section -2, assumption and notation used through the paper are introduced. In section-3 the mathematical model to minimize the total relevant inventory cost under some constraints is developed. In section-4, different cases, arising due to permissible delay period are analysed. In section-5.0 a solution procedure has been developed to find the optimal values of decision variables. In section-6, model is analysed with the help of numerical examples and sensitivity analysis is performed in section-7.0 and some observations are made. Concluding remark is given in the section-8 of the paper.

2. ASSUMPTIONS AND NOTATIONS

The mathematical model of two warehouse inventory model for Non-Instantaneous deteriorating items is based on the following assumptions and notations:

2.1 Assumption

- Replenishment rate is infinite and lead time is zero.
- Shortages are not permitted.
- The time horizon of the inventory system is infinite.
- Goods of OW are consumed only after the consumption of goods stored in RW.
- OW has the limited capacity of storage and RW has unlimited capacity.
- Demand rate is known and constant and given by $f(t) = d$.
- Goods are not deteriorated till a fixed time period. The deteriorated quantity of goods are less than the total demand.
- The unit inventory cost (Holding cost + deterioration cost) in RW is more than that of OW i.e $(h_r - h_w) > c\,(\theta(t) - \alpha)$ where $c > 0$.
- Retailer pays his purchase cost to supplier at the time of ordering for next cycle and earns revenue in terms of interest after sales of goods till the payment is made.
- Goods are instantly transported from RW to retail shop on the basis of continuous release pattern and transportation cost is incurred.
- The items are deteriorated at different rate in both warehouses.

2.2 Notations

O_c: Cost of Ordering per Order.

W: Capacity of OW.

T : The length of replenishment cycle ant time point up to which inventory vanishes in OW.

M: Permissible delay period.

t_w: The point of time up to which inventory does not deteriorated.

t_1: The point of time at which inventory level vanishes in RW .

h_r: The holding cost per unit time in OW.

h_w: The holding cost per unit time in RW.

α : Deterioration rate in RW which is constant and $0 < \alpha < 1$.

$\theta(t)$: Deterioration rate in OW which is time dependent and given by βt where $\beta > 0$

d_c: Deterioration cost per unit of item.

S_p: Selling price per unit of item

I_p : Interest charges per unit of time.

I_e : Interest earned per unit of time.

$I_{r,i}(t)$: The level of inventory in RW at time point t for $i = 1,2$

$I_{w,i}(t)$: The level of inventory in OW at time epoch t for $i = 1,2,3$.

Q: Number of inventory ordered at $t = T$.

Π^i: Revenue earned for cases $i = 1,2,3,4$

$□^i(t_w, t_1, \mathrm{T})$: The present worth total relevant inventory cost per unit time for cases

$i = 1,2,3,4$

$□^m(t_w, t_1, \mathrm{T})$: The present worth optimal relevant inventory cost per unit time.

3. DEVELOPMENT OF MATHEMATICAL MODEL

Initially, a retailer purchase lot size of Q units of items, W units of which is kept into OW and remaining (Q-W) are stocked in RW. (See Figure-1)

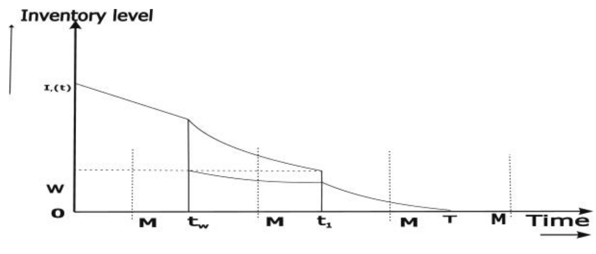


Figure 1.Graphical representation for the inventory model

During the time interval $[0\ t_w]$ the inventory level in RW depleted due to demand only and in this period inventory level in the OW remains W unit. The situation describing the inventory level is governed by the following differential equations:

$$\frac{dI_{r,1}(t)}{dt} = -f(t); \qquad 0 \le t \le t_w \tag{1}$$

$$\frac{dI_{w,1}(t)}{dt} = 0; \qquad 0 \le t \le t_w \tag{2}$$

After time $t = t_w$, in the time interval $[0\ t_1]$ the inventory level in RW depleted due to the combined effect of demand and deterioration and reaches to zero at time point $t = t_1$ and in

OW, inventory level is depleted due to deterioration only. The situation is governed by following differential equations:

$$\frac{dI_{r,2}(t)}{dt} = -\alpha I_r(t) - f(t) \quad ; \quad t_w \le t \le t_1 \qquad (3)$$

$$\frac{dI_{w,2}(t)}{dt} = -\theta(t) I_{w,2}(t) \qquad t_w \le t \le t_1 \qquad (4)$$

Now at time $t = t_1$, when stock out in RW and demand is fulfilled form OW stocks. Stocks in OW in the time interval $[t_1, T]$ depleted due to combined effects of demand and deterioration. Differential equation describing the situation is given as

$$\frac{dI_{w,3}(t)}{dt} = -\theta(t) I_{w,3}(t) - f(t);\ t_1 \le t \le T \qquad (5)$$

Solution of eqs. (1)-(5) with boundary conditions $I_{r,1}(t) = I_{r,2}(t)\ at\ t = t_w$, $I_{r,2}(t) = 0\ at\ t = t_1$, $I_{w,1}(t) = W\ at\ t = 0$, $I_{w,2}(t) = W\ at\ t = t_w$ and $I_{w,3}(t) = 0\ at\ t = T$ are as follows

$$I_{r,1}(t) = D(t_w - t) + \frac{D}{\alpha}(e^{\alpha(t_1 - t_w)} - 1); \qquad 0 \le t \le t_w \qquad (6)$$

$$I_{r,2}(t) = \frac{D}{\alpha}(e^{\alpha(t_1 - t)} - 1); \qquad t_w \le t \le t_1 \qquad (7)$$

$$I_{w,1}(t) = W; \qquad 0 \le t \le t_w \qquad (8)$$

$$I_{w,2}(t) = W\, e^{\frac{\beta}{2}(t_w^2 - t^2)}; \qquad t_w \le t \le t_1 \qquad (9)$$

$$I_{w,3}(t) = D\left((T - t) + \frac{\beta}{6}(T^3 - t^3)\right) e^{-\frac{\beta}{2}t^2};\ t_w \le t \le T \quad (10)$$

Since $at\ t = 0,\ Q - W = I_{r,1}(0)$ which gives

$$Q = W + D\, t_w + \frac{D}{\alpha}(e^{\alpha(t_1 - t_w)} - 1 \qquad (11)$$

Number of inventory deteriorated in RW and OW respectively

$$DI_r = \alpha \int_{t_w}^{t_1} I_{r,2}(t)dt \quad \text{and}$$

$$DI_w = \beta\left(\int_{t_w}^{t_1} I_{w,2}(t)dt + \int_{t_1}^{T} I_{w,3}(t)dt\right)$$

Now present worth of total inventory cost consist of the following components:

1. Present worth ordering cost CO is C_o
2. Present worth of cost in RW, HR is $h_r\left(\int_0^{t_w} I_{r,1}(t)dt + \int_{t_w}^{t_1} I_{r,2}(t)dt\right)$
3. Present worth of holding cost in OW ,HW is

$$h_w\left(\int_0^{t_w} I_{w,1}(t)dt + \int_{t_w}^{t_1} I_{w,2}(t)dt + \int_{t_1}^{T} I_{w,3}(t)dt\right)$$

4. Present worth of deterioration DC is

$$d_c\left\{\alpha \int_{t_w}^{t_1} I_{r,2}(t)dt + \beta\left(\int_{t_w}^{t_1} I_{w,2}(t)dt + \int_{t_1}^{T} I_{w,3}(t)dt\right)\right\}$$

5. Present worth transportation cost TC is

$$t_c \int_0^{t_1} f(t)dt$$

On simplification following are obtained

$$RH = h_r\left\{\frac{dh_r}{2\alpha}\left(\alpha t_w^2 + 2t_w(e^{\alpha(t_1 - t_w)} - 1)\right) + \frac{D}{\alpha^2}\left(\left(e^{\alpha(t_1 - t_w)} - 1\right) - \alpha(t_1 - t_w)\right)\right\}$$

$$HW = h_w\left\{W\, t_w + W\, e^{\frac{\beta}{2}t_w^2}\left((t_1 - t_w) - \frac{\beta}{6}(t_1^3 - t_w^3)\right) + \frac{T^2}{2} + \frac{t_1^2}{2} - t_1 T + \frac{\beta T^4}{12} - \frac{\beta t_1^4}{12} + \frac{\beta T t_1^3}{6} - \frac{\beta t_1 T^3}{6}\right\}$$

$$DC = d_c\left\{\beta W\, e^{\frac{\beta}{2}t_w^2} + d\beta\left(\frac{T^2}{2} + \frac{t_1^2}{2} - t_1 T + \frac{\beta T^4}{12} - \frac{\beta t_1^4}{12} + \frac{\beta T t_1^3}{6} - \frac{\beta t_1 T^3}{6}\right) + \frac{D}{\alpha}\left(\left(e^{\alpha(t_1 - t_w)} - 1\right) - \alpha(t_1 - t_w)\right)\right\}$$

$$TC = t_c\, d\, t_1$$

As M is the permissible delay period for retailer given by supplier, beyond which an interest will be charged by the supplier. As per pictorial representation of inventory level at Figure-1, there may arises the following cases depending upon the parameter values t_μ, t_r, M and T which are discussed separately in the section 4.

4. CASE ANALYSIS

Case-1: $0 < M \le t_w$

Case-2: $t_w < M \le t_1$

Case-3: $t_1 < M \le T$

Case-4: $T < M$

Now each case is discussed separately:

Case-1: $0 < M \le t_w$

In this case two different scenarios may arise depending upon the willingness of the retailer and supplier which are as follows:

Subcase -1.1 If retailer wishes to pay full amount to the supplier at $t = M$ then he does not pay any interest and earn interest form his sales revenue till end of cycle length. Therefore interest earned by the retailer is

$$IE_{1.1} = S_P I_e\, d\, \frac{M^2}{2} + I_e\left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right)(T - M) + S_P I_e\, d\, \frac{(t_w - M)^2}{2} + \left(S_P\, d\, (t_w - M) + S_P I_e\, d\, \frac{(t_w - M)^2}{2}\right) I_e(T - t_w) + S_P I_e\, d\, \frac{(t_1 - t_w)^2}{2} + \left(S_P\, d\, (t_1 - t_w) + S_P I_e\, d\, \frac{(t_1 - t_w)^2}{2}\right) I_e(T - t_1) + S_P I_e\, d\, \frac{(T - t_1)^2}{2}; \qquad (1.1)$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(1.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_1}{T} \qquad (1.2)$$

where

$$\mathrm{Y_1} = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$$

Hence the corresponding optimization problem is

Problem-1. Minimize $\square^{(1.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_1}{T}$ (1.3)

Subject to $0 < M \le t_w < t_1 < T$

Subcase -1.2: If retailer wishes to make partial payment. In this case again two scenarios may arise:

Scenario-1.2.1: Retailer wishes to pay a part of his total purchased cost at $t = M$ and remaining amount

$P_c Q - \left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right)$ at $t = K$ where K > M.

Therefore total amount paid at $t = K$ is

$P_cQ - \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right) + I_e\left(P_cQ - \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right)\right)$
and the revenue earned by the retailer till K is

$$\left(S_P\ d\ (K-M) + S_P I_e\ d\ \frac{(K-M)^2}{2}\right)$$

Now the amount available to retailer = amount payable to supplier at $t = K$ i.e.

$$\left(P_cQ - (R_1)\right) + I_e\left(P_cQ - (R_1)\right) = \left(S_P\ d\ (K-M) + S_P I_e\ d\ \frac{(K-M)^2}{2}\right)$$

where $R_1 = \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right)$

Simplifying above eq. we get

$$S_P I_e\ d\ K^2 - \left((S_P\ d\ M I_e - S_P\ d) + (P_cQ - (R_1))I_p\right)K - (P_cQ - (R_1)) + \left(S_P\ d - (P_cQ - (R_1))I_p\right)M = 0$$

This is a quadratic in K. The admissible solution of K is given by

$$K = M + \frac{-(S_P d - I_P A_1\ ) + \sqrt{(S_P d - I_P A_1\ )^2 - 2A_1 d S_P I_e}}{S_P I_e d}$$

where $A_1 = (P_cQ - (R_1))$ (1.4)

In this case total interest earned by the retailer is

$$IE_{1.2.1} = S_P I_e\ d\ \frac{(T-K)^2}{2}$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(1.2.1)}(t_w, t_1\ , \mathrm{T}) = \frac{Y_2}{T} \quad (1.5)$$

where $\mathrm{Y}_2 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-2.Minimize $\square^{(1.2.1)}(t_w, t_1\ , \mathrm{T}) = \frac{Y_2}{T}$; (1.6)

Subject to $0 < M \le t_w < t_1 < T$

Scenario-1.2.2: If retailer makes full payment after the permissible delay period when possible due to not willingness of supplier for partial payment. Let he pays at $t = K(\mathrm{K} > M)$.Now the total amount paid by retailer at K is P_cQ and interest on P_cQ for period $(K-M)$ i.e. $P_cQ(1 + I_P(K-M))$

The total revenue earned by the retailer up to K is $S_P dK$ and interest on $S_P dK$ for period $(K-M)$ i.e.

$$\left(S_P\ d\ K\left(1 + I_e \frac{K}{2}\right)\right).$$

Obviously the amount payable to supplier is amount available to retailer at K that is

$$P_cQ\left(1 + I_P(K-M)\right) = \left(S_P\ d\ K\left(1 + I_e\frac{K}{2}\right)\right)$$

After simplification above equation reduces to a quadratic equation in K.The admissible solution of K is given by

$$K = \frac{-(S_P d - I_P P_c Q\ ) + \sqrt{(S_P d - I_P P_c Q\ )^2 - 2dS_P I_e P_c Q(1 + M I_P)}}{S_P I_e d}; \quad (1.7)$$

In this case total interest earned by the retailer is

$$IE_{1.2.2} = S_P I_e\ d\ \frac{(T-K)^2}{2}$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(1.2.2)}(t_w, t_1\ , \mathrm{T}) = \frac{Y_3}{T};$$

where $\mathrm{Y}_3 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-3.Minimize $\square^{(1.2.2)}(t_w, t_1\ , \mathrm{T}) = \frac{Y_3}{T}$; (1.8)

Subject to $0 < M \le t_w < t_1 < T$

Case-2.0: $t_w < M \le t_1$

In this case two different scenarios depending upon the willingness of the retailer and supplier may also arise and given as follows:

Subcase -2.1 If retailer wishes to pay full amount to the supplier at $t = M$ then he does not pay any interest and earn interest form his sales revenue till end of cycle length. Therefore interest earned by the retailer is

$$IE_{2.1} = S_P I_e\ d\ \frac{M^2}{2} + I_e\left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right)(T-M) + S_P I_e\ d\ \frac{(t_1 - M)^2}{2} + \left(S_P\ d\ (t_1 - M) + S_P I_e\ d\ \frac{(t_1 - M)^2}{2}\right) I_e (T - t_1) + S_P I_e\ d\ \frac{(T - t_1)^2}{2}; \quad (2.1)$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(2.1)}(t_w, t_1\ , \mathrm{T}) = \frac{Y_4}{T}$$

where $\mathrm{Y}_4 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-4.Minimize $\square^{(2.1)}(t_w, t_1\ , \mathrm{T}) = \frac{Y_4}{T}$; (2.2)

Subject to $0 < t_w < M \le t_1 < T$

Subcase -2.2: If retailer wishes to make partial payment. In this case again two scenarios may arises:

Scenario-2.2.1: Retailer wishes to pay a part of his total purchased cost at $t = M$ and remaining amount $P_cQ - \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right)$ at $t = K$ after M where K > M. Total amount paid at $t = K$ is

$$P_cQ - \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right) + I_e\left(P_cQ - \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right)\right)$$

and also retailer earns interest on his sales revenue till K. The total revenue that retailer earns up to time point K is

$$\left(S_P\ d\ (B-M)\left(1 + I_e\ \frac{(B-M)}{2}\right)\right)$$

Now the amount available to retailer = amount payable to supplier at $t = K$ i.e.

$$\left(P_cQ - (R_1)\right) + I_e\left(P_cQ - (R_1)\right) = \left(S_P\ d\ (K-M) + S_P I_e\ d\ \frac{(K-M)^2}{2}\right)$$

where $R_1 = \left(S_P\ d\ M + S_P I_e\ d\ \frac{M^2}{2}\right)$

Simplifying above eq. we get

$$S_P I_e\, d\, K^2 - \big((S_P\, d\, M I_e - S_P\, d) + (P_c Q - (R_1))I_p\big)K - (P_c Q - (R_1)) + \big(S_P\, d - (P_c Q - (R_1))I_p\big)M = 0$$

This is quadratic in K.The admissible solution of K is given by

$$K = M + \frac{-(S_P d - I_P A_1) + \sqrt{(S_P d - I_P A_1)^2 - 2A_1 d S_P I_e}}{S_P I_e d}; \quad (2.3)$$

In this case total interest earned by the retailer is

$$IE_{2.1.1} = S_P I_e\, d\, \frac{(T-K)^2}{2}$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(2.1.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_5}{T}$$

where $\mathrm{Y}_5 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-5.Minimize $\square^{(2.1.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_5}{T}$; (2.4)

Subject to $0 < t_w < M \le t_1 < T$

Scenario-2.2.2: If retailer makes full payment after the permissible delay period (when possible due to not willingness of supplier for partial payment). Let he pays at $t = K(\mathrm{K} > M)$ .Now the total amount paid by retailer at K is $P_c Q$ and interest on $P_c Q$ for period $(K - M)$ i.e.

$$P_c Q(1 + I_P(K - M))$$

The total revenue earned by the retailer up to K is $S_P dB$ and interest on $S_P dB$ for period $(K - M)$ i.e.

$$\left(S_P\, d\, K\left(1 + I_e \frac{K}{2}\right)\right).$$

Obviously the amount payable to supplier is amount available to retailer at K is

$$P_c Q\big(1 + I_P(K - M)\big) = \left(S_P\, d\, K\left(1 + I_e \frac{K}{2}\right)\right)$$

After simplification above equation reduces to a quadratic equation in K. The admissible solution of K is given by

$$K = \frac{-(S_P d - I_P P_c Q) + \sqrt{(S_P d - I_P P_c Q)^2 - 2 d S_P I_e P_c Q(1 + M I_P)}}{S_P I_e d}; \quad (2.5)$$

In this case total interest earned by the retailer is

$$IE_{2.1.2} = S_P I_e\, d\, \frac{(T-K)^2}{2}$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(2.1.2)}(t_w, t_1, \mathrm{T}) = \frac{Y_6}{T}$$

where $\mathrm{Y}_6 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-6.Minimize $\square^{(2.1.2)}(t_w, t_1, \mathrm{T}) = \frac{Y_6}{T}$; (2.6)

Case-3.0: $t_1 < M \le T$

In this case also two different scenarios may arise depending upon the willingness of the retailer and supplier which are as follows:

Subcase-3.1 If retailer wishes to pay full amount to the supplier at $t = M$ then he does not pay any interest and earn interest form his sales revenue till end of cycle length. Therefore interest earned by the retailer is

$$IE_{3.1} = S_P I_e\, d\, \frac{M^2}{2} + I_e\left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right)(T - M) + S_P I_e\, d\, \frac{(M-t_1)^2}{2} + S_P I_e\, d\, \frac{(T-M)^2}{2}; \quad (3.1)$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(3.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_7}{T}$$

where $\mathrm{Y}_7 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-7.Minimize $\square^{(3.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_7}{T}$ (3.2)

Subject to $0 < t_w < t_1 < M \le T$

Subcase -3.2: If retailer wishes to make partial payment. In this case again two scenarios

may appear:

Scenaio-3.2.1: Retailer wishes to pay a part of his total purchased cost at $t = M$ and remaining amount

$P_c Q - \left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right)$ at $t = K$ after M where K > M. Total amount paid at $t = K$ is

$$P_c Q - \left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right) + I_e\left(P_c Q - \left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right)\right).$$

Also retailer earns interest on his sales revenue till K. The total revenue that retailer earns up to time point K is

$$\left(S_P\, d\,(K - M)\left(1 + I_e\, \frac{(K-M)}{2}\right)\right)$$

Now the amount available to retailer = amount payable to supplier at $t = K$ i.e.

$$\big(P_c Q - (R_1)\big) + I_e\big(P_c Q - (R_1)\big) = \left(S_P\, d\,(B - M) + S_P I_e\, d\, \frac{(B-M)^2}{2}\right)$$

where $R_1 = \left(S_P\, d\, M + S_P I_e\, d\, \frac{M^2}{2}\right)$

Simplifying above eq. we get

$$S_P I_e\, d\, K^2 - \big((S_P\, d\, M I_e - S_P\, d) + (P_c Q - (R_1))I_p\big)K - (P_c Q - (R_1)) + \big(S_P\, d - (P_c Q - (R_1))I_p\big)M = 0$$

This is quadratic in K.The admissible solution of K is given by

$$K = M + \frac{-(S_P d - I_P A_1) + \sqrt{(S_P d - I_P A_1)^2 - 2A_1 d S_P I_e}}{S_P I_e d}; \quad (3.3)$$

In this case total interest earned by the retailer is

$$IE_{3.2.1} = S_P I_e\, d\, \frac{(T-K)^2}{2}$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$$\square^{(3.2.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_8}{T}$$

Subject to $0 < t_w < M \le t_1 < T$

where $\mathrm{Y}_8 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned}$

Hence the corresponding optimization problem is

Problem-8.Minimize $\square^{(3.2.1)}(t_w, t_1, \mathrm{T}) = \frac{Y_8}{T}$; (3.4)

Subject to $0 < t_w < t_1 < M \leq T$

Scenario-3.2.2: If retailer makes full payment after the permissible delay period when possible due to not willingness of supplier for partial payment. Let he pays total purchase cost at $t = K(\mathrm{K} > M)$ .Now the total amount paid by retailer at K is $P_c Q$ and interest on it for the period $(K - M)$ i.e.

$P_c Q(1 + I_P(K - M))$

The total revenue earned by the retailer up to K is $S_P dB$ and interest on $S_P dK$ for period $(K - M)$ i.e.

$\left(S_P\ d\ K\left(1 + I_e \frac{K}{2}\right)\right)$.

Obviously the amount payable to supplier is amount available to retailer at K that is

$$P_c Q\big(1 + I_P(K - M)\big) = \left(S_P\ d\ K\left(1 + I_e \frac{K}{2}\right)\right)$$

After simplification above equation reduces to a quadratic equation in K. The admissible solution of K is given by

$$K = \frac{-(S_P d - I_P P_c Q\ ) + \sqrt{(S_P d - I_P P_c Q\ )^2 - 2 d S_P I_e P_c Q(1 + M I_P)}}{S_P I_e d}; \qquad (3.5)$$

In this case total interest earned by the retailer is

$IE_{3.2.2} = S_P I_e\ d\ \frac{(T-K)^2}{2}$

Therefore, the relevant inventory cost per unit of time for the cycle is given by

$\square^{(3.2.2)}(t_w, t_1, \mathrm{T}) = \frac{Y_9}{T}$

where $\mathrm{Y}_5 = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned}$

Hence the corresponding optimization problem is

Problem-9.Minimize $\square^{(3.2.2)}(t_w, t_1, \mathrm{T}) = \frac{Y_9}{T}$; (3.6)

Subject to $0 < t_w < t_1 < M \leq T$

Case-4.0: $T < M$

In this case also retailer has not to pay any interest charged and accumulate interest on revenue collected from the sales, therefore

$$IE_{4.0} = S_P I_e\ d\ \frac{T^2}{2} + I_e\left(S_P\ d\ T + S_P I_e\ d\ \frac{T^2}{2}\right)(M - T)$$

Therefore, the relevant inventory cost per unit of time for the cycle is given by $\square^{(4.0)}(t_w, t_1, \mathrm{T}) = \frac{Y_{10}}{T}$

where $\mathrm{Y}_{10} = \lfloor \mathrm{CO} + \mathrm{HR} + \mathrm{HW} + \mathrm{DC} + \mathrm{TC} + \text{Interest paid} - \text{Interest earned} \rfloor$

Hence the corresponding optimization problem is

Problem-10.Minimize $\square^{(4.0)}(t_w, t_1, \mathrm{T}) = \frac{Y_{10}}{T}$; (4.1)

Subject to $0 < t_w < t_1 < M \leq T$

5. SOLUTION ALGORITHM

Summarizing the above arguments, the following solution procedure is established to find the optimal solution.

Solution procedure

Step 0: Input all the initial value of parameters.

Step 1: If retailer pay full amount at t =M then solve the constrained optimization problem (i.e. problem-1) for case-1.1 and store the result as $t_w^{1.1}, t_1^{1.1}, T^{1.1}, Q^{1.1}$, and $\square^{1.1}$ else go to Step-2.

Step 2: If partial payment is made at t=M. then solve the constrained optimization problem (i.e. problem-2) for case-1.2.1 and store the result as $t_w^{1.2.1}, t_1^{1.2.1}, T^{1.2.1}, Q^{1.2.1}$, and $\square^{1.2.1}$ else go to Step-3.

Step 3: Solve the constrained optimization problem (i.e. problem-3) for case-1.2.2 and store the result as $t_w^{1.2.2}, t_1^{1.2.2}$, $T^{1.2.2}, Q^{1.2.2}$, and $\square^{1.2.2}$.

Step 4: Find the optimal solution for case-1.2 from the solutions of case-1.2.1 and Case-1.2.2.Hence denote the optimal inventory cost per unit of time as $\square^{1.2} = \min\{\square^{1.2.1}, \square^{1.2.2}\}$ and denote the corresponding values of t_w, t_1, T, and Q as $t_w^{1.2}, t_1^{1.2}, T^{1.2}$, and $Q^{1.2.2}$.

Step 5: The optimal solution of case-1 can be determined from the solutions of case-1.1 and Case-1.2. Hence for case-1 the optimal inventory cost per unit of time is given by $\square^1 = \min\{\square^{1.1}, \square^{1.2}\}$ and the corresponding values of t_w, t_1, T, and Q as t_w^1, t_1^1, T^1, and Q^1.

Proceeding in the similar way, the problems of other cases can be solved. The optimal total inventory cost for case-2, case-3, case-4 and the corresponding solutions of decision variables and ordered quantity are denoted as $\square^2$, $\square^3$, $\square^4$ (t_w^2, t_1^2, T^2, Q^2), (t_w^3, t_1^3, T^3, Q^3) and (t_w^4, t_1^4, T^4, Q^4) respectively.

The optimal solution of the inventory system can be determined by comparing the total relevant inventory cost for all the cases. Hence the optimal total relevant inventory cost per unit of time is given by $\square^* = \min\{\square^1, \square^2, \square^3, \square^4\}$.The corresponding values of optimal decision variables and ordered quantity for the problem is denoted by (t_w^*, t_1^*, T^*, Q^*).

6. NUMERICAL EXAMPLES

In order to illustrate the above model with the help of above solution procedure, we consider the following examples:

Parameter	C_o	d	W	d_c	P_c	S_p	I_p	I_e	β	α	h_r	h_w	t_c	M
Example-1	1500	400	100	10	10	15	0.15	0.12	0.10	0.06	20	15	0.1	0.75
Example-2	1500	600	300	10	10	15	0.15	0.12	0.10	0.06	10	8	0.1	0.75

Table 1

Case	Subcase	Scenario	t_w	t_1	T	Q	Average I.C.	Remarks
1	1.1	-	9.1967	12.8007	15.2886	5220	30450.80	
	1.2	1.2.1	9.4194	13.1492	15.4981	5360	35448.20	
		1.2.2	8.8950	12.5551	15.5818	5122	36681.80	
2	2.1		9.2485	12.9323	15.5055	5269	31099.32	
	2.2	2.2.1	9.7430	13.6565	16.2339	5563	37675.60	
		2.2.2	10.1843	14.2337	16.6362	5793	41834.00	
3	3.1		9.3364	13.0262	15.1038	5310	33202.90	
	3.2	3.2.1	20.1561	27.5523	33.5635	11121	18298.20	
		3.2.2	-	-	-	-	-	Infeasible
4	-	-	6.9586	9.9207	13.1923	4068	12733.20	

Table 2

Case	Subcase	Scenario	t_w	t_1	T	Q	Average I.C.	Remarks
1	1.1		6.5040	9.2066	11.6143	5823.96	14642.90	
	1.2	1.2.1	6.7389	9.6356	11.9075	6081.36	19103.10	
		1.2.2	6.7286	9.8205	12.8370	6192.30	22231.70	
2	2.1		6.5071	9.3351	11.8627	5901.06	14769.10	
	2.2	2.2.1	7.3890	10.6366	12.9380	6681.96	21206.40	
		2.2.2	6.6859	9.7684	12.8141	6161.04	22374.40	
3	3.1		6.6695	9.5455	11.2658	6027.00	17833.40	
	3.2	3.2.1	7.3631	10.6043	12.9646	6662.58	21034.70	
		3.2.2	6.9675	9.6758	12.7710	6105.48	22667.00	
4	-	-	-	-	-	-	-	Infeasible

Note: Results of examples 1 and 2 are listed in Table 1 & Table 2 respectively.

6.1 Numerical Analysis

1. From Table-1, it is observed that in case-4, total relevant inventory cost is minimum. Also the ordering cycle length and the order quantity are lower as compared to other cases when the permissible delay period is more than the ordering cycle length. From Table-2, it is observed that the total relevant inventory cost, ordering cycle length and ordered quantity are lower than the other cases when retailer pay full amount to the supplier at the end of permissible delay period.
2. Fixing the value of T^*, the convexity of the optimal inventory cost with respect to optimal t_w^* and t_1^* in each case is depicted in Figure-2 with the help of 3-D graphs. Because of high non-linearity of the function; convexity of the model cannot be tested analytically. The convexity of the graph shows that the solution under constrained is unique and global one.

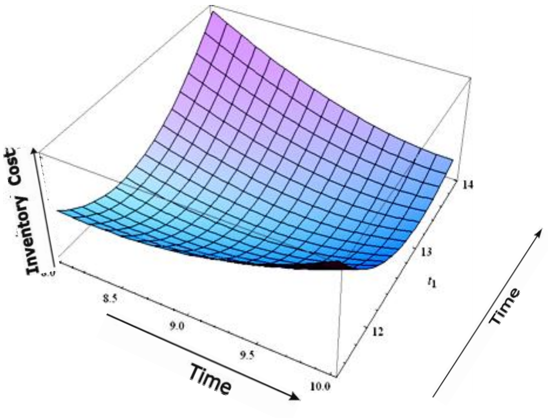
7. SENSITIVITY ANALYSIS

Considering example-1 mentioned in section 7.0, sensitivity analysis is performed to study the effect of changes of the parameters on the optimal policy and the results are given in Table-3.

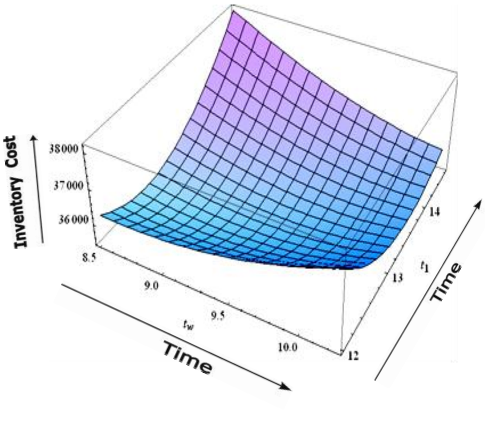
7.1 Observations

From Table-3, the following observations can be made:

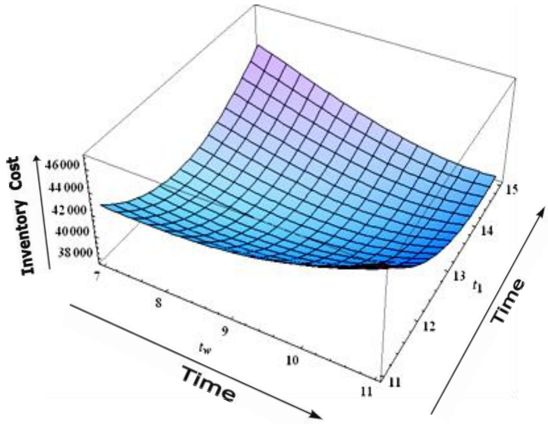
1) The total relevant inventory cost is sensitive to the demand, holding cost in RW, deterioration rate in RW ordering cost, permissible delay period and transportation cost and increases with increment of these parameters value.
2) The total relevant inventory cost is highly sensitive to the selling price of the products and earned interest rate as it increases the total relevant inventory cost decreases e.g. about 30% increase in selling price yields approximately 50% decrease in total relevant inventory cost.
3) The total relevant inventory cost is sensitive to the Capacity of OW, deterioration rate in OW and holding cost in OW.As the value of these parameters are increases the total relevant inventory cost decreases.
4) The increase in the value of interest paid does not affect the total relevant inventory cost of the model.



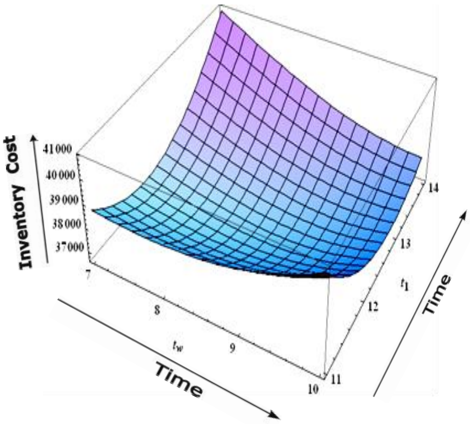
Case-1.1



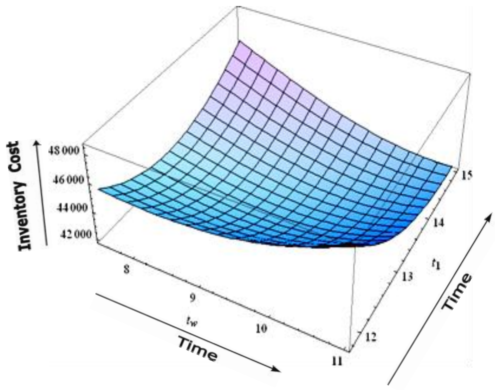
Case-1.2.1



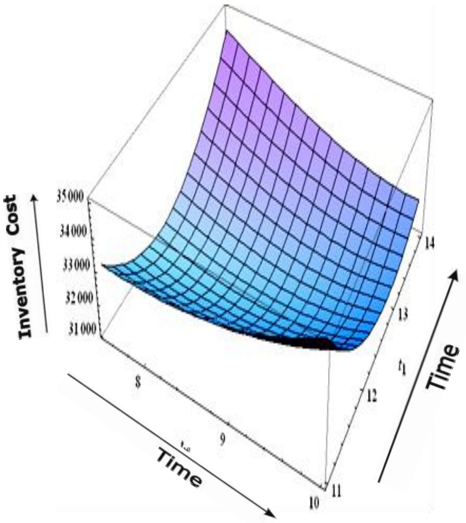
Case-2.1



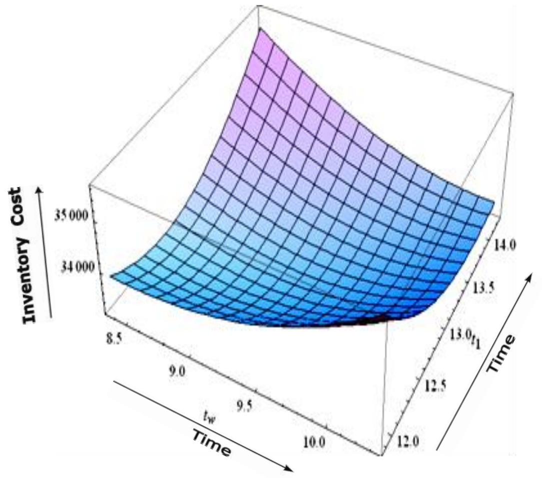
Case-1.2.2



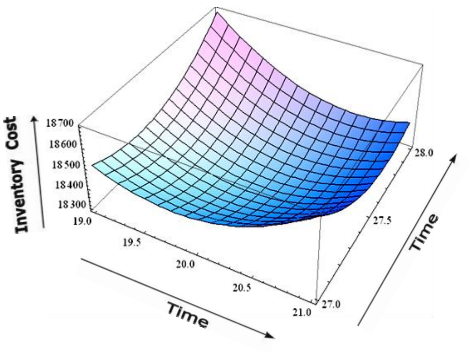
Case-2.2.2



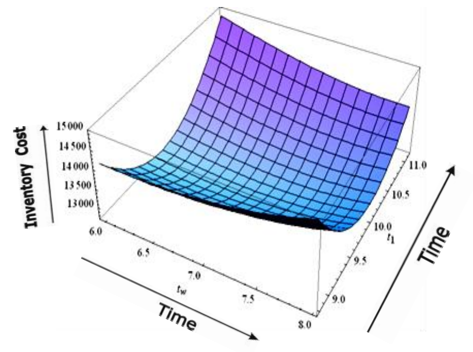
Case-2.2.1



Case-3.1



Case-3.2.1



Case-4.0

Figure-2: Graph representing inventory cost function vs. t_w^* and t_1^*

Table-3

Sensitivity in parameters changed for example-1

Parameter	change value	t_w	t_1	T	$Ordered\ quantity$	$Average\ I.cost$	$Remarks$
d (400)	600	7.9348	11.1386	14.5309	6783.16	20683.50	case-4.0
C_o (1500)	2000	6.9537	9.9143	13.1868	4065.72	12771.10	case-4.0
S_p (15)	20	5.4818	8.0183	11.4297	3307.00	6378.31	case-4.0
P_c (10)	15	9.0591	12.5458	14.9399	5118.32	30113.60	Case-1.1
h_r (20)	25	8.1923	11.5322	15.1576	4712.00	20879.60	Case-4.0
h_w (15)	20	6.3854	9.2311	11.9331	4712.88	13609.30	Case-4.0
W (100)	200	5.4232	8.8348	11.1735	3733.92	12081.60	Case-4.0
I_p (0.15)	0.20	6.9586	9.9207	13.1923	4068.00	12733.20	Case-4.0
I_e (0.12)	0.15	5.1412	7.5843	10.9964	3133.72	6180.71	Case-4.0
d_c (10)	15	9.0591	12.5458	14.9399	5118.32	30113.60	Case-4.0
M 0.75	2.0	7.0492	10.0384	13.2936	4115.00	14348.30	Case-4.0
α (0.06)	0.08	7.0005	9.9250	13.1978	4070.00	12751.50	Case-4.0
β (0.10)	0.15	6.9265	9.7023	11.6471	3980.00	23598.80	Case-1.1

8. CONCLUDING REMARKS

In this paper, we proposed a deterministic two-warehouse inventory model for non-instantaneous deteriorating items with constant demand and permissible delay period in payment under assumption that items are transported from RW to retails shop under continuous release pattern with the objective of minimizing the total relevant inventory cost function of the inventory model. We see that total relevant inventory cost is influenced by selling price and earned interest rate. The total relevant inventory cost is found to be minimum when the permissible delay period is larger than the ordering cycle length or when retailer pays his total purchase cost at the end of permissible delay period. Furthermore the proposed model can be used in inventory control of certain non- instantaneous deteriorating items and can be further extended by incorporating time dependent demand, probabilistic demand pattern and variable holding cost etc.

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