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On monotonicity of some combinatorial sequences

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Abstract. We confirm Sun's conjecture that $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n\geqslant 4}$ is strictly decreasing to the limit 1, where $(F_n)_{n\geqslant 0}$ is the Fibonacci sequence. We also prove that the sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n\geqslant 3}$ is strictly decreasing with limit 1, where D_n is the n-th derangement number. For m-th order harmonic numbers $H_n^{(m)}=\sum_{k=1}^n 1/k^m$ $(n=1,2,3,\ldots)$, we show that $\left(\sqrt[n+1]{H_{n+1}^{(m)}}/\sqrt[n]{H_n^{(m)}}\right)_{n\geqslant 3}$ is strictly increasing.

1. Introduction

A challenging conjecture of Firoozbakht states that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for every } n=1,2,3,\ldots,$$

where p_n denotes the n-th prime. Note that $\lim_{n\to\infty}\sqrt[n]{p_n}=1$ by the Prime Number Theorem. In [4] the second author conjectured further that for any integer $n>4$ we have the inequality

$$\frac{\sqrt[n+1]{p_{n+1}}}{\sqrt[n]{p_n}} < 1-\frac{\log\log n}{2n^2},$$

which has been verified for all $n\leqslant 3.5\times 10^6$. Motivated by this and [3], Sun [4, Conjecture 2.12] conjectured that the sequence $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 7}$ is strictly

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increasing, where S_n is the sum of the first n positive squarefree numbers. Moreover, he also posed many conjectures on monotonicity of sequences of the type $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n\geqslant N}$ with $(a_n)_{n\geqslant 1}$ a familiar combinatorial sequence of positive integers.

Throughout this paper, we set $\mathbb{N}=\{0,1,2,\ldots\}$ and $\mathbb{Z}^+=\{1,2,3,\ldots\}$.

Let A and B be integers with $\Delta=A^2-4B\neq 0$. The Lucas sequence $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ is defined as follows:

$$u_0=0,\ u_1=1,\quad \text{and}\quad u_{n+1}=Au_n-Bu_{n-1} \text{ for } n=1,2,3,\ldots.$$

It is well known that $u_n=(\alpha^n-\beta^n)/(\alpha-\beta)$ for all $n\in\mathbb{N}$, where

$$\alpha=\frac{A+\sqrt{\Delta}}{2}\quad \text{and}\quad \beta=\frac{A-\sqrt{\Delta}}{2}$$

are the two roots of the characteristic equation $x^2-Ax+B=0$. The sequence $F_n=u_n(1,-1)$ $(n\in\mathbb{N})$ is the famous Fibonacci sequence, see [1, p. 46] for combinatorial interpretations of Fibonacci numbers.

Our first result is as follows.

Theorem 1.1. *Let $A>0$ and $B\neq 0$ be integers with $\Delta=A^2-4B>0$, and set $u_n=u_n(A,B)$ for $n\in\mathbb{N}$. Then there exists an integer $N>0$ such that the sequence $(\sqrt[n+1]{u_{n+1}}/\sqrt[n]{u_n})_{n\geqslant N}$ is strictly decreasing with limit 1. In the case $A=1$ and $B=-1$ we may take $N=4$.*

Remark 1.1. Under the condition of Theorem 1.1, by [2, Lemma 4] we have $u_n<u_{n+1}$ unless $A=n=1$. Note that the second assertion in Theorem 1.1 confirms a conjecture of the second author [4, Conjecture 3.1] on the Fibonacci sequence.

For $n\in\mathbb{Z}^+$ the n-th derangement number D_n denotes the number of permutations σ of $\{1,\ldots,n\}$ with $\sigma(i)=i$ for no $i=1,\ldots,n$. It has the following explicit expression (cf. [1, p. 67]):

$$D_n=\sum_{k=0}^{n}(-1)^k\frac{n!}{k!}.$$

Our second theorem is the following result conjectured by the second author [4, Conjecture 3.3].

Theorem 1.2. *The sequence $\left(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n}\right)_{n\geqslant 3}$ is strictly decreasing with limit 1.*

Remark 1.2. It follows from Theorem 1.2 that the sequence $(\sqrt[n]{D_n})_{n\geqslant 2}$ is strictly increasing.

For each $m \in \mathbb{Z}^+$ those $H_n^{(m)} = \sum_{k=1}^n 1/k^m$ $(n \in \mathbb{Z}^+)$ are called harmonic numbers of order m. The usual harmonic numbers (of order 1) are those rational numbers $H_n = H_n^{(1)}$ $(n = 1, 2, 3, \ldots)$.

Our following theorem confirms Conjecture 2.16 of Sun [4].

Theorem 1.3. *For any $m \in \mathbb{Z}^+$, the sequence $\left(\sqrt[n+1]{H_{n+1}^{(m)}} \,/\, \sqrt[n]{H_n^{(m)}}\right)_{n\geqslant 3}$ is strictly increasing.*

We will prove Theorems 1.1–1.3 in Sections 2–4 respectively. It seems that there is no simple form for the generating function $\sum_{n=0}^\infty \sqrt[n]{a_n}\, x^n$ with $a_n = u_n, D_n, H_n^{(m)}$. Note also that the set of those sequences $(a_n)_{n\geqslant 1}$ of positive numbers with $\left(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n}\right)_{n\geqslant 1}$ decreasing (or increasing) is closed under multiplication.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Set

$$\alpha = \frac{A+\sqrt{\Delta}}{2}, \ \beta = \frac{A-\sqrt{\Delta}}{2}, \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{A-\sqrt{\Delta}}{A+\sqrt{\Delta}}.$$

Then

$$\log u_n = \log \frac{\alpha^n(1-\gamma^n)}{\alpha-\beta} = n\log\alpha + \log(1-\gamma^n) - \log\sqrt{\Delta}$$

for any $n \in \mathbb{Z}^+$. Note that

$$\log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} = \frac{\log(1-\gamma^{n+1})}{n+1} - \frac{\log(1-\gamma^n)}{n} + \frac{\log\sqrt{\Delta}}{n(n+1)}.$$

Since

$$\lim_{n\to\infty} \frac{\log(1-\gamma^n)}{n} = \lim_{n\to\infty} \frac{-\gamma^n}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n(n+1)} = 0,$$

we deduce that

$$\lim_{n\to\infty} \log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 0, \quad \text{i.e.,} \ \lim_{n\to\infty} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 1.$$

For any $n \in \mathbb{Z}^+$, clearly

$$\frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} > \frac{\sqrt[n+2]{u_{n+2}}}{\sqrt[n+1]{u_{n+1}}} \iff \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} > \frac{\log u_{n+2}}{n+2} - \frac{\log u_{n+1}}{n+1}$$
$$\iff \Delta_n := \frac{2\log u_{n+1}}{n+1} - \frac{\log u_n}{n} - \frac{\log u_{n+2}}{n+2} > 0.$$

Observe that

$$\begin{aligned}\Delta_n &= 2\log\alpha + \frac{2\log(1-\gamma^{n+1})}{n+1} - \frac{2\log\sqrt{\Delta}}{n+1}\\ &\quad - \left(2\log\alpha + \frac{\log(1-\gamma^n)}{n} + \frac{\log(1-\gamma^{n+2})}{n+2} - \frac{\log\sqrt{\Delta}}{n} - \frac{\log\sqrt{\Delta}}{n+2}\right)\\ &= \frac{\log\Delta}{n(n+1)(n+2)} + \frac{2}{n+1}\log(1-\gamma^{n+1}) - \frac{\log(1-\gamma^n)}{n} - \frac{\log(1-\gamma^{n+2})}{n+2}.\end{aligned}$$

The function $f(x) = \log(1+x)$ on the interval $(-1,+\infty)$ is concave since $f''(x) = -1/(x+1)^2 < 0$. Note that $|\gamma| < 1$. If $-|\gamma| \leqslant x \leqslant 0$, then $t = -x/|\gamma| \in [0,1]$ and hence

$$f(x) = f(t(-|\gamma|) + (1-t)0) \geqslant tf(-|\gamma|) + (1-t)f(0) = qx,$$

where $q = -\log(1-|\gamma|)/|\gamma| > 0$. Note also that $\log(1+x) < x$ for $x > 0$. So we have

$$\begin{aligned}&\log\left(1-\gamma^{n+1}\right) \geqslant \log\left(1-|\gamma|^{n+1}\right) \geqslant -q|\gamma|^{n+1},\\ &\log\left(1-\gamma^{n}\right) \leqslant \log\left(1+|\gamma|^{n}\right) < |\gamma|^{n},\\ &\log\left(1-\gamma^{n+2}\right) \leqslant \log\left(1+|\gamma|^{n+2}\right) < |\gamma|^{n+2}.\end{aligned}$$

Therefore

$$\Delta_n > \frac{\log\Delta}{n(n+1)(n+2)} - |\gamma|^n\left(\frac{2q|\gamma|}{n+1} + \frac{1}{n} + \frac{|\gamma|^2}{n+2}\right)$$

and hence

$$\begin{aligned}&n(n+1)(n+2)\Delta_n\\ &\qquad > \log\Delta - |\gamma|^n\left(2q|\gamma|n(n+2) + (n+1)(n+2) + |\gamma|^2 n(n+1)\right). \quad (1)\end{aligned}$$

Since $\lim_{n\to\infty} n^2|\gamma|^n = 0$, when $\Delta > 1$ we have $\Delta_n > 0$ for large n.

Now it remains to consider the case $\Delta = 1$. Clearly $\gamma = (A-1)/(A+1) > 0$. Recall that $\log(1-x) < -x$ for $x \in (0,1)$. As

$$\frac{d}{dx}(\log(1-x) + x + x^2) = -\frac{1}{1-x} + 1 + 2x = \frac{x(1-2x)}{1-x} > 0 \quad \text{for } x \in (0, 0.5),$$

we have $\log(1-x)+x+x^2 > \log 1 + 0 + 0^2 = 0$ for $x \in (0, 0.5)$. If n is large enough, then $\gamma^n < 0.5$ and hence

$$\Delta_n = \frac{2}{n+1}\log(1-\gamma^{n+1}) - \frac{\log(1-\gamma^n)}{n} - \frac{\log(1-\gamma^{n+2})}{n+2} > w_n,$$

where

$$w_n := \frac{2}{n+1}(-\gamma^{n+1}-\gamma^{2n+2}) + \frac{\gamma^n}{n} + \frac{\gamma^{n+2}}{n+2}.$$

Note that

$$\lim_{n\to\infty}\frac{nw_n}{\gamma^n} = -2\gamma+1+\gamma^2 = (1-\gamma)^2 > 0.$$

So, for sufficiently large n we have $\Delta_n > w_n > 0$.

Now we show that $n \geqslant 4$ suffices in the case $A = 1$ and $B = -1$. Note that $\Delta = 5$ and $\gamma \approx -0.382$. The sequence $(|\gamma|^n(n+1)(n+2))_{n\geqslant 1}$ is decreasing since

$$|\gamma|\frac{(n+2)(n+3)}{(n+1)(n+2)} < \frac{1}{2}\left(1+\frac{2}{n+1}\right) \leqslant 1$$

for $n \geqslant 1$. It follows that $|\gamma|^n(n+1)(n+2) \leqslant \gamma^6 \times 7 \times 8 < 1/3$ for $n \geqslant 6$. In view of (1), if $n \geqslant 6$ then

$$\begin{aligned} n(n+1)(n+2)\Delta_n &> \log 5 - |\gamma|^n(n+1)(n+2)\left(2q|\gamma|+1+|\gamma|^2\right) \\ &> \log 5 - \frac{1+1+\gamma^2}{3} > \log 5 - 1 > 0. \end{aligned}$$

It is easy to verify that Δ_4 and Δ_5 are positive. So $\left(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n}\right)_{n\geqslant 4}$ is strictly decreasing.

In view of the above, we have completed the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $n \geqslant 3$. It is well known that $|D_n - n!/e| \leqslant 1/2$ (cf. [1, p. 67]). Applying the Intermediate Value Theorem in calculus, we obtain

$$\left|\log D_n - \log\left(\frac{n!}{e}\right)\right| \leqslant \left|D_n - \frac{n!}{e}\right| \leqslant 0.5.$$

Set $R_0(n) = \log D_n - \log n!$. Then $|R_0(n)| \leqslant 1.5$.

Since $\lim_{n\to\infty} R_0(n)/n = 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\log D_{n+1}}{n+1}-\frac{\log D_n}{n}\right) &= \lim_{n\to\infty}\left(\frac{\log((n+1)!)}{n+1}-\frac{\log(n!)}{n}\right)\\ &= \lim_{n\to\infty}\frac{n\log(n+1)+n\log(n!)-(n+1)\log(n!)}{n(n+1)}\\ &= \lim_{n\to\infty}\frac{n\log n+n\log(1+1/n)-\log(n!)}{n(n+1)}\\ &= \lim_{n\to\infty}\frac{\log(n^n/n!)}{n(n+1)}.\end{aligned}$$

As $n! \sim \sqrt{2\pi n}(n/e)^n$ (i.e., $\lim_{n\to\infty} n!/(\sqrt{2\pi n}(n/e)^n) = 1$) by Stirling's formula, we have $\log(n^n/n!) \sim n$ and hence

$$\lim_{n\to\infty}\left(\frac{\log D_{n+1}}{n+1}-\frac{\log D_n}{n}\right)=0.$$

Thus $\lim_{n\to\infty} \sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n} = 1$.

From the known identity $D_n/n! = \sum_{k=0}^{n}(-1)^k/k!$, we have the recurrence $D_n = nD_{n-1} + (-1)^n$ for $n > 1$. Thus, if $n \geqslant 3$ then

$$R_0(n)-R_0(n-1)=\log\frac{D_n}{n!}-\log\frac{D_{n-1}}{(n-1)!}=\log\frac{D_n}{nD_{n-1}}=\log\left(1+\frac{(-1)^n}{nD_{n-1}}\right).$$

Fix $n \geqslant 4$. If n is even, then

$$0<R_0(n)-R_0(n-1)=\log\left(1+\frac{1}{nD_{n-1}}\right)<\frac{1}{nD_{n-1}}=\frac{1}{D_n-1}\leqslant\frac{3}{D_n+0.5}.$$

If n is odd, then

$$0>R_0(n)-R_0(n-1)=\log\left(1-\frac{1}{nD_{n-1}}\right)>\frac{-2}{nD_{n-1}}=\frac{-2}{D_n+1}\geqslant\frac{-3}{D_n+0.5}$$

since $\log(1-x)+2x>0$ for $x\in(0,0.5)$. So

$$|R_0(n)-R_0(n-1)|<\frac{3}{D_n+0.5}\leqslant\frac{3e}{n!}$$

and hence

$$\left|\frac{R_0(n-1)-R_0(n)}{n-1}\right|<\frac{3e}{n!(n-1)}\leqslant\frac{3e}{n(n-1)(n+1)}.$$

Similarly, we also have

$$\left|\frac{R_0(n+1)-R_0(n)}{n+1}\right| < \frac{3e}{n!(n+1)} \leqslant \frac{3e}{n(n-1)(n+1)}.$$

Therefore,

$$\begin{aligned}&\left|\frac{R_0(n+1)}{n+1}-\frac{2R_0(n)}{n}+\frac{R_0(n-1)}{n-1}-\frac{2R_0(n)}{n(n-1)(n+1)}\right|\\ &=\left|\frac{R_0(n+1)-R_0(n)}{n+1}+\frac{R_0(n-1)-R_0(n)}{n-1}\right| \leqslant \frac{6e}{n(n-1)(n+1)}\end{aligned}$$

and hence

$$\left|\frac{R_0(n+1)}{n+1}-\frac{2R_0(n)}{n}+\frac{R_0(n-1)}{n-1}\right| \leqslant \frac{2|R_0(n)|+6e}{n(n-1)(n+1)} \leqslant \frac{6e+3}{n(n-1)(n+1)}.$$

Thus $|R_1(n)| \leqslant 6e+3$, where

$$R_1(n) := n(n-1)(n+1)\left(\frac{R_0(n+1)}{n+1}-\frac{2R_0(n)}{n}+\frac{R_0(n-1)}{n-1}\right).$$

Since

$$\begin{aligned}\log((n-1)!) &= \sum_{k=1}^{n-1}\int_k^{k+1}(\log k)dx < \sum_{k=1}^{n-1}\int_k^{k+1}\log x dx\\ &= \int_1^n \log x dx = n\log n - n + 1 < \sum_{k=1}^{n-1}\int_k^{k+1}(\log(k+1))dx = \log(n!),\end{aligned}$$

we have

$$n\log n - n < \log(n!) = \log((n-1)!) + \log n < n\log n - n + \log n + 1$$

and so $\log(n!) = n\log n - n + R_2(n)$ with $|R_2(n)| < \log n + 1$.

Observe that

$$\begin{aligned}&\frac{\log D_{n+1}}{n+1}-\frac{2}{n}\log D_n+\frac{\log D_{n-1}}{n-1}\\ &\quad=\frac{\log((n+1)!)}{n+1}-\frac{2\log(n!)}{n}+\frac{\log((n-1)!)}{n-1}+\frac{R_1(n)}{(n-1)n(n+1)}\\ &\quad=\frac{2\log(n!)}{(n-1)n(n+1)}-\frac{\log n}{n-1}+\frac{\log(n+1)}{n+1}+\frac{R_1(n)}{(n-1)n(n+1)}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2n}{(n-1)n(n+1)} + \frac{\log(n+1)-\log n}{n+1} + \frac{2R_2(n)+R_1(n)}{(n-1)n(n+1)} \\
&\leqslant -\frac{2n}{(n-1)n(n+1)} + \frac{n-1}{(n-1)n(n+1)} + \frac{2R_2(n)+R_1(n)}{(n-1)n(n+1)} \\
&= -\frac{n+1-2R_2(n)-R_1(n)}{(n-1)n(n+1)}.
\end{aligned}
$$

If $n \geqslant 27$, then $n+1-2R_2(n)-R_1(n) > n - 2\log n - 1 - 6e - 3 > 0$, and hence we get

$$\log \frac{\sqrt[n]{D_n}}{\sqrt[n-1]{D_{n-1}}} > \log \frac{\sqrt[n+1]{D_{n+1}}}{\sqrt[n]{D_n}}.$$

By a direct check via computer, the last inequality also holds for $n = 4, \ldots, 26$. Therefore, the sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n\geqslant 3}$ is strictly decreasing. This ends the proof. □

4. Proof of Theorem 1.3

Lemma 4.1. *For $x > 0$ we have*

$$\log(1+x) > x - \frac{x^2}{2}. \tag{2}$$

Proof. As

$$\frac{d}{dx}\left(\log(1+x) - x + \frac{x^2}{2}\right) = \frac{x^2}{1+x},$$

we see that $\log(1+x) - x + x^2/2 > \log 1 - 0 + 0^2/2 = 0$ for any $x > 0$. □

Lemma 4.2. *Let $m, n \in \mathbb{Z}^+$ with $n \geqslant 3$. If $m \geqslant 11$ or $n \geqslant 30$, then*

$$H_n^{(m)} \log H_n^{(m)} > 4\left(\frac{2}{n+2}\right)^{m-1}. \tag{3}$$

Proof. Recall that H_n refers to $H_n^{(1)}$. If $n \geqslant 30$, then

$$H_n \log H_n \geqslant H_{30} \log H_{30} > 4,$$

and hence (3) holds for $m = 1$.

Below we assume that $m \geqslant 2$. As $n \geqslant 3$, we have

$$H_n^{(m)} \log H_n^{(m)} \geqslant H_3^{(m)} \log H_3^{(m)}.$$

So it suffices to show that

$$\left(\frac{n+2}{2}\right)^{m-1} H_3^{(m)} \log H_3^{(m)} > 4 \tag{4}$$

whenever $m \geqslant 11$ or $n \geqslant 30$. By Lemma 4.1,

$$\begin{aligned}\log H_3^{(m)} &= \log(1+2^{-m}+3^{-m}) > 2^{-m}+3^{-m} - \frac{(2^{-m}+3^{-m})^2}{2}\\ &> 2^{-m}+3^{-m} - \frac{(2^{1-m})^2}{2} = \frac{1}{2^m} + \frac{1}{3^m} - \frac{2}{4^m}.\end{aligned}$$

If $m \geqslant 3$, then $(4/3)^m \geqslant (4/3)^3 > 2$ and hence $\log H_3^{(m)} > 1/2^m$. Note also that $H_3^{(2)} \log H_3^{(2)} > 1/4$. So we always have

$$H_3^{(m)} \log H_3^{(m)} > \frac{1}{2^m}.$$

If $m \geqslant 11$, then $1.25^m \geqslant 1.25^{11} > 10$ and hence

$$\frac{1}{2^m} > \frac{4}{2.5^{m-1}} \geqslant \frac{4}{((n+2)/2)^{m-1}},$$

therefore (4) holds. When $n \geqslant 30$, we have

$$\frac{1}{2^m} \geqslant \frac{1}{2^{4m-6}} = \frac{4}{16^{m-1}} \geqslant 4\left(\frac{2}{n+2}\right)^{m-1}$$

and hence (4) also holds. □

Proof of Theorem 1.3. Let $m \geqslant 1$ and $n \geqslant 3$. Set

$$\Delta_n(m) := \log \frac{\sqrt[n+1]{H_{n+1}^{(m)}}}{\sqrt[n]{H_n^{(m)}}} - \log \frac{\sqrt[n+2]{H_{n+2}^{(m)}}}{\sqrt[n+1]{H_{n+1}^{(m)}}} = \frac{2\log H_{n+1}^{(m)}}{n+1} - \frac{\log H_n^{(m)}}{n} - \frac{\log H_{n+2}^{(m)}}{n+2}.$$

It suffices to show that $\Delta_n(m) < 0$. This can be easily verified by computer if $m \in \{1,\dots,10\}$ and $n \in \{3,\dots,29\}$.

Below we assume that $m \geqslant 11$ or $n \geqslant 30$. Recall (2) and the known fact that $\log(1+x) < x$ for $x > 0$. We clearly have

$$\log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} = \log\left(1 + \frac{1}{(n+1)^m H_n^{(m)}}\right) < \frac{1}{(n+1)^m H_n^{(m)}}$$

and

$$\log \frac{H_{n+2}^{(m)}}{H_n^{(m)}} > \log\left(1+\frac{2}{(n+2)^m H_n^{(m)}}\right) > \frac{2}{(n+2)^m H_n^{(m)}} - \frac{2}{(n+2)^{2m}(H_n^{(m)})^2}.$$

It follows that

$$\begin{aligned}\Delta_n(m) &= \left(\frac{2}{n+1}-\frac{1}{n}-\frac{1}{n+2}\right)\log H_n^{(m)} + \frac{2}{n+1}\log\frac{H_{n+1}^{(m)}}{H_n^{(m)}} - \frac{1}{n+2}\log\frac{H_{n+2}^{(m)}}{H_n^{(m)}} \\ &< \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2}{(n+1)^{m+1}H_n^{(m)}} \\ &\quad - \frac{2}{(n+2)^{m+1}H_n^{(m)}} + \frac{2}{(n+2)^{2m+1}(H_n^{(m)})^2}.\end{aligned}$$

Since $(n+2)^{m+1} = \sum_{k=0}^{m+1}\binom{m+1}{k}(n+1)^k$ by the binomial theorem, we obtain

$$\begin{aligned}\Delta_n(m) &\leqslant \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2\sum_{k=0}^{m}\binom{m+1}{k}(n+1)^k}{(n+1)^{m+1}(n+2)^{m+1}H_n^{(m)}} + \frac{2}{(n+2)^{m+2}H_n^{(m)}} \\ &< \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(n+1)^m\sum_{k=0}^{m}\binom{m+1}{k}}{(n+1)^{m+1}(n+2)^{m+1}H_n^{(m)}} + \frac{2}{(n+1)(n+2)^{m+1}H_n^{(m)}} \\ &= \frac{-2\log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(2^{m+1}-1)+2}{(n+1)(n+2)^{m+1}H_n^{(m)}}.\end{aligned}$$

Thus

$$\begin{aligned}n(n+1)(n+2)\Delta_n(m)\frac{H_n^{(m)}}{2} &< -H_n^{(m)}\log H_n^{(m)} + \frac{2^{m+1}n}{(n+2)^m} \\ &< 4\left(\frac{2}{n+2}\right)^{m-1} - H_n^{(m)}\log H_n^{(m)}.\end{aligned}$$

Applying (3) we find that $\Delta_n(m) < 0$. This completes the proof. □

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