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Existence of solutions for coupled integral boundary value problem at resonance

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Abstract. A coupled integral boundary value problem for a nonlinear differential system is considered in this article. An existence result is obtained with the use of the coincidence degree theory.

1. Introduction

This article deals with the following second order coupled integral boundary value problem

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t), x'(t), y'(t)), \quad t \in (0,1), \\ -y''(t) = f_2(t, x(t), y(t), x'(t), y'(t)), \quad t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x], \end{cases} \tag{1}$$

where f_1 and $f_2 : (0,1) \times \mathbb{R}^4 \to \mathbb{R}$ are continuous and may be singular at $t = 0, 1$; $\alpha[x], \beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\alpha[x] = \int_0^1 x(t)dA(t), \qquad \beta[x] = \int_0^1 x(t)dB(t),$$

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involving Stieltjes integrals.

The coupled integral boundary value problem (1) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} -x''(t) = 0, t \in (0,1), \\ -y''(t) = 0, t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \alpha[y], \quad y(1) = \beta[x] \end{cases}$$

has nontrivial solutions. Clearly, the resonance condition is $\kappa_1\kappa_2 = 1$, where

$$\kappa_1 = \int_0^1 t dA(t), \qquad \kappa_2 = \int_0^1 t dB(t).$$

Coupled boundary conditions for ordinary differential systems arise in the study of reaction-diffusion equations and Sturm–Liouville problems, and have wide applications in various fields of sciences and engineering, for example mathematical biology and heat equation. Moreover, boundary value problems with Riemann–Stieltjes integral conditions constitute a very interesting and important class of problems. They include two, three, multi-point and integral boundary-value problems as special cases, see [10], [11], [16]. The existence and multiplicity of solutions for such problems have received a growing attention in the literature. We refer the reader to [1], [3], [4], [12], [13], [14], [16] for some recent results of integral boundary value problems at nonresonance and to [2], [4], [7], [8], [9], [15], [17], [18] at resonance. However, to our knowledge, the existence of solutions for a differential system with coupled integral boundary value problems at resonance has not been studied.

The purpose of this paper is to study the existence of solution for coupled integral boundary value problems (1) at resonance. Our method is based upon the coincidence degree theory of Mawhin [5], [6].

The organization of this paper is as follows. In Section 2, we provide some necessary background. In particular, we shall introduce some lemmas and definitions associated with problem (1). In Section 3, the main results of problem (1) will be stated and proved. Finally, one example is also included to illustrate the main results.

Throughout this paper, we always suppose that

$$(H) \quad \kappa_1\kappa_2 = 1, \qquad \kappa = \frac{\kappa_1}{2}\int_0^1 t(1-t)dB(t) + \frac{1}{2}\int_0^1 t(1-t)dA(t) \neq 0.$$

2. Preliminaries

In this section, to establish the existence of solutions in $C^1[0,1] \times C^1[0,1]$, we provide some background definitions and lemmas.

Definition 2.1. Let Y, Z be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator. L is said to be the Fredholm operator of index zero provided that:

(i) $\operatorname{Im} L$ is a closed subset of Z,

(ii) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$.

Let Y, Z be real Banach spaces and $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm operator of index zero. $P : Y \to Y$, $Q : Z \to Z$ are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$ and $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible. We denote the inverse of the operator by K_P (generalized inverse operator of L). If Ω is an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \emptyset$, the operator $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

The theorem we use is Theorem 2.4 of [5] or Theorem IV.13 [6].

Theorem 2.1. *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$.*

(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.*

(iii) *$\deg(QN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0) \neq 0$, where $Q : Z \to Z$ is a projector as above with $\operatorname{Im} L = \operatorname{Ker} Q$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

We use the classical spaces $C[0,1]$, $C^1[0,1]$ and $L^1[0,1]$. For $x \in C^1[0,1]$, we use the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max\limits_{t \in [0,1]} |x(t)|$. And denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We also use the following three Banach spaces:

$$W^{2,1}(0,1) = \{x : [0,1] \to \mathbb{R} \mid x, x' \text{ are absolutely cont. on } [0,1],\ x'' \in L^1[0,1]\}$$

with its usual norm and $Y = C^1[0,1] \times C^1[0,1]$, with the norm

$$\|(x,y)\|_Y = \max\{\|x\|, \|y\|\} = \max\{\|(x,y)\|_\infty, \|(x',y')\|_\infty\}$$

and $Z = L^1[0,1] \times L^1[0,1]$, with the norm

$$\|(x,y)\|_Z = \max\{\|x\|_1, \|y\|_1\},$$

where

$$\|(x,y)\|_\infty = \max\{\|x\|_\infty, \|y\|_\infty\}.$$

Let the linear operator $L : \operatorname{dom} L \subset Y \to Z$ with

$$\operatorname{dom} L = \{(x,y) \in W^{2,1}(0,1) \times W^{2,1}(0,1) : x(0) = y(0) = 0,\\ x(1) = \alpha[y],\ y(1) = \beta[x]\}$$

be defined by

$$L(x,y) = (-x'', -y'').$$

Let the nonlinear operator $N : Y \to Z$ be defined by

$$(N(x,y))(t) = (N_1(x,y)(t), N_2(x,y)(t)),$$

where $N_1, N_2 : Y \to L^1[0,1]$ are defined by

$$N_1(x,y)(t) = f_1(t, x(t), y(t), x'(t), y'(t)),\\ N_2(x,y)(t) = f_2(t, x(t), y(t), x'(t), y'(t)).$$

Then coupled integral boundary value problems (1) can be written as

$$L(x,y) = N(x,y).$$

Lemma 2.1. *Let L be the linear operator defined as above. Then*

$$\operatorname{Ker} L = \{(x,y) \in \operatorname{dom}\ L :\ (x,y) = c(\kappa_1 t, t),\ c \in \mathbb{R},\ t \in [0,1]\}$$

and

$$\operatorname{Im} L = \left\{(u,v) \in Z : \kappa_1 \int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) \right.\\ \left. + \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) = 0\right\},$$

where

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

PROOF. Let $(x(t), y(t)) = (\kappa_1 t, t)$. Considering $\kappa_1\kappa_2 = 1$,

$$\alpha[y] = \alpha[t] = \kappa_1 = x(1), \qquad \beta[x] = \beta[\kappa_1 t] = \kappa_1\kappa_2 = y(1).$$

So $\{(x,y) \in \operatorname{dom} L : (x,y) = c(\kappa_1 t, t),\ c \in \mathbb{R}, t \in [0,1]\} \subset \operatorname{Ker} L$. If $L(x,y) = (-x'', -y'') = (0,0)$ and $(x(0), y(0)) = (0,0)$, then $(x(t), y(t)) = (at, bt)$. Considering $x(1) = \alpha[y]$ and $y(1) = \beta[x]$, we can obtain that $a = \alpha[bt] = b\kappa_1$ and $b = \beta[at] = a\kappa_2$. It yields

$$\operatorname{Ker} L \subset \{(x,y) \in \operatorname{dom}\ L : (x,y) = c(\kappa_1 t, t), \quad c \in \mathbb{R},\ t \in [0,1]\}.$$

We now show that

$$\begin{aligned}\operatorname{Im} L = \Bigg\{(u,v) \in Z : \kappa_1 &\int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) \\ &+ \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) = 0\Bigg\}.\end{aligned}$$

If $(u,v) \in \operatorname{Im} L$, then there exists $(x,y) \in \operatorname{dom} L$ for which $-x''(t) = u(t)$ and $-y''(t) = v(t)$. Hence

$$x(t) = \int_0^1 k(t,s)u(s)ds + x(1)t. \tag{2}$$

and

$$y(t) = \int_0^1 k(t,s)v(s)ds + y(1)t. \tag{3}$$

Integrating (2) and (3) with respect to $dB(t)$ and $dA(t)$ respectively on $[0,1]$ gives

$$\int_0^1 x(t)dB(t) = \int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) + x(1)\kappa_2$$

and

$$\int_0^1 y(t)dA(t) = \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) + y(1)\kappa_1.$$

Therefore

$$\begin{bmatrix} -\kappa_2 & 1 \\ 1 & -\kappa_1 \end{bmatrix} \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) \\ \displaystyle\int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) \end{bmatrix}$$

and so

$$-\frac{\kappa_2}{1} = -\frac{1}{\kappa_1} = \frac{\int_0^1 \int_0^1 k(t,s)u(s)dsdB(t)}{\int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)}.$$

It yields

$$\begin{aligned}\operatorname{Im} L \subset \Big\{ (u,v) \in Z : \kappa_1 \int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) \\ + \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) = 0 \Big\}.\end{aligned}$$

On the other hand, $(u,v) \in Z$ satisfies

$$\kappa_1 \int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) + \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) = 0.$$

Let

$$\begin{aligned}x(t) &= \int_0^1 k(t,s)u(s)ds + t,\\ y(t) &= \int_0^1 k(t,s)v(s)ds + \kappa_2 t \left(1 - \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)\right),\end{aligned}$$

then $L(x,y) = (u,v)$, $x(0) = 0$, $y(0) = 0$, $x(1) = 1$ and

$$y(1) = \kappa_2 \left(1 - \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)\right).$$

Simple computations yield

$$\begin{aligned}\int_0^1 x(t)dB(t) &= \int_0^1 \int_0^1 k(t,s)u(s)dsdB(t) + \int_0^1 tdB(t)\\ &= -\kappa_2 \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t) + \kappa_2 = y(1)\end{aligned}$$

and

$$\begin{aligned}\int_0^1 y(t)dA(t) = &\int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)\\ &+ \kappa_2 \int_0^1 tdA(t) \left(1 - \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)\right)\\ = &\int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)\\ &+ \kappa_2\kappa_1 \left(1 - \int_0^1 \int_0^1 k(t,s)v(s)dsdA(t)\right) = 1 = x(1). \qquad \square\end{aligned}$$

Lemma 2.2. *The operator L is a Fredholm operator of index zero and* $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. *Furthermore, the linear operator* $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be defined by*

$$(K_p(x,y))(t) = \left(\int_0^1 k(t,s)x(s)ds + t \int_0^1 \int_0^1 k(t,s)y(s)dsdA(t)\ ,\ \int_0^1 k(t,s)y(s)ds \right).$$

Also

$$\|K_p(x,y)\|_Y \le \triangle \|(x,y)\|_Z, \quad \text{for all } (x,y) \in \operatorname{Im} L,$$

where

$$\triangle = 1 + \int_0^1 t(1-t)d\left(\bigvee_0^t B \right)$$

and $\bigvee_0^t B$ *denotes the variation of* B *on* $[0,t]$.

Proof. Firstly, we construct the following operator $Q : Z \to Z$ by

$$Q(x,y) = \frac{1}{\kappa} \left(\kappa_1 \int_0^1 \int_0^1 k(t,s)x(s)dsdB(t) + \int_0^1 \int_0^1 k(t,s)y(s)dsdA(t) \right)(1,1).$$

Note that $\int_0^1 k(t,s)ds = \frac{1}{2}t(1-t)$, we have

$$Q^2(x,y) = Q(x,y).$$

Thus $Q : Z \to Z$ is a well-defined projector.

Now, it is obvious that $\operatorname{Im} L = \operatorname{Ker} Q$. Noting that Q is a linear projector, we have $Z = \operatorname{Im} Q \oplus \operatorname{Ker} Q$. Hence $Z = \operatorname{Im} Q \oplus \operatorname{Im} L$ and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L$ =1. This means that L is a Fredholm operator of index zero.

Taking $P : Y \to Y$ as

$$(P(x,y))(t) = y(1)(\kappa_1 t, t),$$

then the generalized inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of L can be rewritten

$$(K_p(x,y))(t) = \left(\int_0^1 k(t,s)x(s)ds + t \int_0^1 \int_0^1 k(t,s)y(s)dsdA(t)\ ,\ \int_0^1 k(t,s)y(s)ds \right).$$

In fact, for $(x,y)\in \operatorname{Im} L$, we have

$$(LK_p(x,y))(t)=\Bigg(-\Big(\int_0^1 k(t,s)x(s)ds+t\int_0^1\int_0^1 k(t,s)y(s)dsdA(t)\Big)'',\\ -\Big(\int_0^1 k(t,s)y(s)ds\Big)''\Bigg)=(x(t),y(t))$$

and for $(x,y)\in \operatorname{dom} L\cap \operatorname{Ker} P$, we know

$$\begin{aligned}(K_pL(x,y))(t)&=\Big(-\int_0^1 k(t,s)x''(s)ds-t\int_0^1\int_0^1 k(t,s)y''(s)dsdA(t),\\ &\qquad -\int_0^1\int_0^1 k(t,s)y''(s)ds\Big)\\ &=\Big(x(t)-x(0)(1-t)-x(1)t+t\int_0^1(y(t)-y(0)(1-t)\\ &\qquad -y(1)t)dA(t),\quad y(t)-y(0)(1-t)-y(1)t\Big).\end{aligned}$$

In view of $(x,y)\in \operatorname{dom} L\cap \operatorname{Ker} P$, $x(0)=y(0)=y(1)=0, x(1)=\int_0^1 y(t)dA(t)$, thus

$$(K_pL(x,y))(t)=(x(t),y(t)).$$

This shows that $K_P=(L|_{\operatorname{dom}\cap \operatorname{Ker} P})^{-1}$.

Since

$$\begin{aligned}&\|K_p(x,y)\|_\infty\\ &\quad\le \max\left\{\int_0^1|x(s)|ds+\int_0^1\int_0^1 t(1-t)|y(s)|dsd\left(\bigvee_0^t B\right),\int_0^1|y(s)|ds\right\}\\ &\quad\le \triangle\|(x,y)\|_Z\end{aligned}$$

and

$$\begin{aligned}&\|(K_p(x,y))'\|_\infty\\ &\quad\le \max\left\{\int_0^1|x(s)|ds+\int_0^1\int_0^1 t(1-t)|y(s)|dsd\left(\bigvee_0^t B\right),\int_0^1|y(s)|ds\right\}\\ &\quad\le \triangle\|(x,y)\|_Z.\end{aligned}$$

Thus $\|K_p(x,y)\|_Y\le \triangle\|(x,y)\|_Z$. □

3. Main results

In this section, we will use Theorem 2.1 to prove the existence of solutions to BVP (1). To obtain our main theorem, we use the following assumptions.

(H1) There exist functions $a_i, b_i, c_i, d_i, e_i \in L^1[0,1] (i = 1, 2)$, such that for all $(u, v, u', v') \in \mathbb{R}^4$ and $t \in [0,1]$,

$$|f_1(t, u, v, u', v')| \leq a_1(t)|u| + b_1(t)|v| + c_1(t)|u'| + d_1(t)|v'| + e_1(t);$$

$$|f_2(t, u, v, u', v')| \leq a_2(t)|u| + b_2(t)|v| + c_2(t)|u'| + d_2(t)|v'| + e_2(t);$$

(H2) There exists a constant $C > 0$ such that for $(x, y) \in \operatorname{dom} L$, if $|y'(t)| > C$ for all $t \in [0,1]$, then

$$\begin{aligned}\kappa_1 &\int_0^1 \int_0^1 k(t,s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\ &+ \int_0^1 \int_0^1 k(t,s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) \neq 0;\end{aligned}$$

(H3) There exists a constant $D > 0$ such that for $a \in \mathbb{R}$, if $|a| > D$, then either

$$\begin{aligned}a\kappa_1 &\int_0^1 \int_0^1 k(t,s) f_1(s, \kappa_1 as, as, \kappa_1 a, a) ds dB(t) \\ &+ a \int_0^1 \int_0^1 k(t,s) f_2(s, \kappa_1 as, as, \kappa_1 a, a) ds dA(t) > 0;\end{aligned}$$

or

$$\begin{aligned}a\kappa_1 &\int_0^1 \int_0^1 k(t,s) f_1(s, \kappa_1 as, as, \kappa_1 a, a) ds dB(t) \\ &+ a \int_0^1 \int_0^1 k(t,s) f_2(s, \kappa_1 as, as, \kappa_1 a, a) ds dA(t) < 0.\end{aligned}$$

Theorem 3.1. *Let* (H1)–(H3) *hold. Then* (1) *has at least one solution in* $C^1[0,1] \times C^1[0,1]$ *provided*

$$\max\{\triangle\alpha_1 + \delta\alpha_2, \alpha_2(\delta + \triangle)\} < 1,$$

where $\delta = \max\{|\kappa_1|, 1\}$, $\alpha_i = \|a_i\|_1 + \|b_i\|_1 + \|c_i\|_1 + \|d_i\|_1$ $(i = 1, 2)$ *and* $\triangle$ *is the same as* Lemma 2.2.

Proof. Set

$$\Omega_1 = \{(x, y) \in \operatorname{dom} L \backslash \operatorname{Ker} L : L(x, y) = \lambda N(x, y) \text{ for some } \lambda \in [0, 1]\}.$$

Then, for $(x, y) \in \Omega_1$, $L(x, y) = \lambda N(x, y)$, thus $\lambda \neq 0$, $N(x, y) \in \operatorname{Im} L = \operatorname{Ker} Q$, and hence

$$QN(x, y) = (0, 0), \quad \text{for all } t \in [0, 1].$$

By the definition of Q, we have

$$\begin{aligned}\kappa_1 \int_0^1 \int_0^1 & k(t, s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\ & + \int_0^1 \int_0^1 k(t, s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) = 0.\end{aligned}$$

Thus, from (H2), there exist $t_0 \in [0, 1]$ such that $|y'(t_0)| \leq C$. Since x, x', y, y' are absolutely continuous for all $t \in [0, 1]$,

$$\begin{aligned}|y'(t)| &= |y'(t_0) - \int_{t_0}^t y''(s) ds| \leq |y'(t_0)| + \|y''\|_1 \leq C + \|N_2(x, y)\|_1, \\ |y(t)| &= |y(0) - \int_0^t y'(s) ds| \leq t(C + \|N_2(x, y)\|_1) \leq C + \|N_2(x, y)\|_1.\end{aligned}$$

Thus

$$\begin{aligned}\|P(x, y)\|_Y &= \max\{\|P(x, y)\|_\infty, \|(P(x, y))'\|_\infty\} \\ &\leq \delta |y(1)| \leq \delta(C + \|N_2(x, y)\|_1). \end{aligned} \tag{4}$$

Also for $(x, y) \in \Omega_1$, $(x, y) \in \operatorname{dom} L \backslash \operatorname{Ker} L$, then

$$(I - P)(x, y) \in \operatorname{dom} L \cap \operatorname{Ker} P, \ LP(x, y) = (0, 0),$$

thus from Lemma 2.2, we have

$$\begin{aligned}\|(I - P)(x, y)\|_Y &= \|K_P L(I - P)(x, y)\|_Y \leq \triangle \|L(I - P)(x, y)\|_Z \\ &= \triangle \|L(x, y)\|_Z \leq \triangle \|N(x, y)\|_Z. \end{aligned} \tag{5}$$

From (4) and (5), we obtain

$$\begin{aligned}\|(x, y)\|_Y &= \|P(x, y) + (I - P)(x, y)\|_Y \leq \|P(x, y)\|_Y + \|(I - P)(x, y)\|_Y \\ &\leq \delta C + \delta \|N_2(x, y)\|_1 + \triangle \|N(x, y)\|_Z\end{aligned}$$

$$= \max\{\delta C + \delta\|N_2(x,y)\|_1 + \triangle\|N_1(x,y)\|_1,$$
$$\delta C + (\delta + \triangle)\|N_2(x,y)\|_1\}. \tag{6}$$

Writing $x(t) = \int_0^t x'(s)ds$, we obtain

$$\|x\|_\infty \le \|x\|_1 \le \|x'\|_\infty. \tag{7}$$

Similarly,

$$\|y\|_\infty \le \|y\|_1 \le \|y'\|_\infty. \tag{8}$$

From (6), we discuss various cases.

Case 1. $\|(x,y)\|_Y \le \delta C + (\delta + \triangle)\|N_2(x,y)\|_1$.

From (H1), (7) and (8), we have

$$\|N_2(x,y)\|_1 \le (\|a_2\|_1 + \|c_2\|_1)\|x'\|_\infty + (\|b_2\|_1 + \|d_2\|_1)\|y'\|_\infty + \|e_2\|_1.$$

Consequently, for

$$\|x'\|_\infty, \|y'\|_\infty \le \|(x,y)\|_Y,$$

so,

$$\|x'\|_\infty \le \frac{(\delta + \triangle)((\|b_2\|_1 + \|d_2\|_1)\|y'\|_\infty + \|e_2\|_1) + \delta C}{1 - (\delta + \triangle)(\|a_2\|_1 + \|c_2\|_1)},$$
$$\|y'\|_\infty \le \frac{\|e_2\|_1(\delta + \triangle) + \delta C}{1 - (\delta + \triangle)(\|a_2\|_1 + \|c_2\|_1 + \|b_2\|_1 + \|d_2\|_1)}.$$

Therefore, there exists $M > 0$ such that

$$\|x'\|_\infty < M, \ \|y'\|_\infty < M,$$

so that $\|(x,y)\|_Y < M$. We have shown that Ω_1 is bounded.

Case 2. $\|(x,y)\|_Y \le \delta C + \delta\|N_2(x,y)\|_1 + \triangle\|N_1(x,y)\|_1$.

From (H1), (7) and (8), we have

$$\|N_1(x,y)\|_1 \le (\|a_1\|_1 + \|c_1\|_1)\|x'\|_\infty + (\|b_1\|_1 + \|d_1\|_1)\|y'\|_\infty + \|e_1\|_1,$$
$$\|N_2(x,y)\|_1 \le (\|a_2\|_1 + \|c_2\|_1)\|x'\|_\infty + (\|b_2\|_1 + \|d_2\|_1)\|y'\|_\infty + \|e_2\|_1,$$

$$\|(x,y)\|_Y \le \delta C + (\triangle(\|a_1\|_1 + \|c_1\|_1) + \delta(\|a_2\|_1 + \|c_2\|_1))\|x'\|_\infty$$
$$+ (\triangle(\|b_1\|_1 + \|d_1\|_1) + \delta(\|b_2\|_1 + \|d_2\|_1))\|y'\|_\infty + \triangle\|e_1\|_1 + \delta\|e_2\|_1,$$

$$\|x'\|_\infty \le \frac{\delta C + (\triangle(\|b_1\|_1 + \|d_1\|_1) + \delta(\|b_2\|_1 + \|d_2\|_1))\|y'\|_\infty + \triangle\|e_1\|_1 + \delta\|e_2\|_1}{1 - (\triangle(\|a_1\|_1 + \|c_1\|_1) + \delta(\|a_2\|_1 + \|c_2\|_1))},$$
$$\|y'\|_\infty \le \frac{\delta C + \triangle\|e_1\|_1 + \delta\|e_2\|_1}{1 - \triangle\alpha_1 - \delta\alpha_2}.$$

From the above inequality, there exists a constant $M > 0$ such that

$$\|x'\|_\infty < M, \ \|y'\|_\infty < M.$$

Therefore Ω_1 is bounded.

Let

$$\Omega_2 = \{(x, y) \in \operatorname{Ker} L : N(x, y) \in \operatorname{Im} L\}.$$

For $(x, y) \in \Omega_2$, $(x, y) \in \operatorname{Ker} L$ implies that (x, y) can be defined by $(x, y) = c(\kappa_1 t, t)$, $t \in [0, 1]$, $c \in \mathbb{R}$. By (H2), there exist $t_0 \in [0, 1]$ such that $|y'(t_0)| \le C$, then

$$\|y'\|_\infty = |c| \le C, \quad \|x'\|_\infty = |c\kappa_1| \le \max\{\kappa_1, 1\}C.$$

Moreover,

$$\|x\|_\infty \le \|x'\|_\infty, \quad \|y\|_\infty \le \|y'\|_\infty.$$

So $\|(x, y)\|_Y \le \max\{\kappa_1, 1\}C$. Thus, Ω_2 is bounded.

We define the isomorphism $J : \operatorname{Ker} L \to \operatorname{Im} Q$ by

$$J(a\kappa_1 t, at) = (a, a).$$

If the first part of (H3) is satisfied, and then let

$$\Omega_3 = \{(x, y) \in \operatorname{Ker} L : \lambda J(x, y) + (1 - \lambda)QN(x, y) = (0, 0), \ \lambda \in [0, 1]\}.$$

For every $(x, y) = a(\kappa_1 t, t) \in \Omega_3$,

$$\lambda(a, a) = -\frac{1-\lambda}{\kappa}\left(\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 as, as, \kappa_1 a, a) ds dB(t) \right.$$
$$\left. + \int_0^1 \int_0^1 k(t, s) f_2(s, \kappa_1 as, as, \kappa_1 a, a) ds dA(t)\right)(1, 1).$$

If $\lambda = 1$, then $a = 0$, and if $|a| > D$, then by (H3)

$$\lambda(a^2, a^2) = -\frac{1-\lambda}{\kappa}\left(a\kappa_1 \int_0^1 \int_0^1 k(t, s) f_1(s, \kappa_1 as, as, \kappa_1 a, a) ds dB(t)\right.$$

$$+a\int_0^1\int_0^1 k(t,s)f_2(s,\kappa_1 as, as, \kappa_1 a, a)dsdA(t)\Big)(1,1)<(0,0),$$

which, in either case, is a contradiction. If the other part of (H3) is satisfied, then we take

$$\Omega_3=\{(x,y)\in \operatorname{Ker}L:-\lambda J(x,y)+(1-\lambda)QN(x,y)=(0,0),\ \lambda\in[0,1]\},$$

and, again, obtain a contradiction. Thus, in either case

$$\|(x,y)\|_Y=\max\{\|x\|,\|y\|\}=\max\{\kappa_1,1\}|a|\le\max\{\kappa_1,1\}D$$

for all $(x,y)\in\Omega_3$, that is, Ω_3 is bounded. In the following, we shall prove that all the conditions of Theorem 2.1 are satisfied.

Set Ω be a bounded open subset of Y such that $\cup_{i=1}^3\overline{\Omega_i}\subset\Omega$. By using the Ascoli–Arzela theorem, we can prove that $K_P(I-Q)N:\Omega\to Y$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by the above argument we have

(i) $Lx\neq\lambda Nx$, for every $(x,\lambda)\in[(\operatorname{dom}L\setminus\operatorname{Ker}L)\cap\partial\Omega]\times(0,1)$,

(ii) $Nx\notin\operatorname{Im}L$ for $x\in\operatorname{Ker}L\cap\partial\Omega$.

At last we will prove that (iii) of Theorem 2.1 is satisfied. Let $H((x,y),\lambda)=\pm\lambda J(x,y)+(1-\lambda)QN(x,y)$. According to above argument, we know

$$H((x,y),\lambda)\neq0\quad\text{for }(x,y)\in\operatorname{Ker}L\cap\partial\Omega,$$

thus, by the homotopy property of degree

$$\begin{aligned}\deg(QN|_{\operatorname{Ker}L},\operatorname{Ker}L\cap\Omega,0)&=\deg(H(\cdot,0),\operatorname{Ker}L\cap\Omega,0)\\&=\deg(H(\cdot,1),\operatorname{Ker}L\cap\Omega,0)\\&=\deg(\pm J,\operatorname{Ker}L\cap\Omega,0)\neq0.\end{aligned}$$

Then by Theorem 2.1, $L(x,y)=N(x,y)$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$, and so, the BVP (1) has at least one solution in the space Y. □

4. Example

To illustrate how our main results can be used in practice, we present an example. Consider the couple boundary value problem

$$\begin{cases}-x''(t)=\dfrac{1}{12}\sin x(t)+\sin y^{\frac13}(t)+e^t+\dfrac16 y'(t),\quad t\in(0,1),\\[2mm]-y''(t)=1+\cos t+\cos y(t)\sin x'(t)+\dfrac18(1+t)y'(t),\quad t\in(0,1),\\[2mm]x(0)=y(0)=0,\quad x(1)=\dfrac45\displaystyle\int_0^1 y(s)ds,\quad y(1)=-x(\frac12)+\frac92x(\frac23).\end{cases}$$

Let

$$f_1(t,x,y,x',y') = \frac{1}{12}\sin x + \sin y^{\frac{1}{3}} + e^t + \frac{1}{6}y',$$
$$f_2(t,x,y,x',y') = 1 + \cos t + \cos y \sin x' + \frac{1}{8}(1+t)y',$$

$$A(t) = \frac{4}{5}t, \quad B(t) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right), \\ -1 & t \in \left[\frac{1}{2}, \frac{2}{3}\right), \\ \frac{7}{2} & t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Then

$$|f_1(t,x,y,x',y')| \le \frac{1}{12}|x| + \frac{1}{6}|y'| + 1 + e, \quad |f_2(t,x,y)| \le \frac{1}{8}(1+t)|y'| + 3,$$

$$\kappa_1 = \frac{2}{5}, \quad \kappa_2 = \frac{5}{2}, \quad \kappa = \frac{13}{60}, \quad \delta = 1, \quad \triangle = \frac{9}{4}.$$

Again taking $b_1 = c_1 = a_2 = b_2 = c_2 = 0$, $a_1 = \frac{1}{12}$, $d_1 = \frac{1}{6}$ and $d_2 = \frac{1}{8}(1+t)$, we have

$$\max\{\triangle(\|a_1\|_1 + \|b_1\|_1 + \|c_1\|_1 + \|d_1\|_1) + \delta(\|a_2\|_1 + \|b_2\|_1 + \|c_2\|_1 + \|d_2\|_1),$$
$$(\delta + \triangle)(\|a_2\|_1 + \|b_2\|_1 + \|c_2\|_1 + \|d_2\|_1)\} = \frac{39}{64} < 1.$$

Finally taking $C = 30$, for any $y \in W^{2,1}(0,1)$, assume $|y'(t)| > C$ hold for any $t \in (0,1)$, since the continuity of y', then either $y'(t) > C$ or $y'(t) < -C$ hold for any $t \in (0,1)$.

If $y'(t) > C$ hold for any $t \in (0,1)$, then

$$f_1(t,x(t),y(t),x'(t),y'(t)) > 0, \quad f_2(t,x(t),y(t),x'(t),y'(t)) > 0.$$

Therefore,

$$\kappa_1 \int_0^1 \int_0^1 k(t,s) f_1(s,x(s),y(s),x'(s),y'(s)) ds dB(t)$$
$$+ \int_0^1 \int_0^1 k(t,s) f_2(s,x(s),y(s),x'(s),y'(s)) ds dA(t)$$
$$> \kappa_1 \int_0^1 \int_0^1 k(t,s) f_1(s,x(s),x'(s))\, ds\, dB(t)$$

$$
\begin{aligned}
&= \kappa_1 \left(- \int_0^1 k(\frac{1}{2}, s) f_1(s, x(s), x'(s)) ds + \frac{9}{2} \int_0^1 k(\frac{2}{3}, s) f_1(s, x(s), x'(s)) ds \right) \\
&\geq \kappa_1 \left(\frac{9}{2} \int_0^1 \frac{2}{3} (1 - \frac{2}{3}) s(1-s) f_1(s, x(s), x'(s)) ds \right. \\
&\quad \left. - \int_0^1 s(1-s) f_1(s, x(s), x'(s)) ds \right) = 0.
\end{aligned}
$$

If $y'(t) < -C$ hold for any $t \in (0,1)$, then

$$
\begin{aligned}
&\kappa_1 \int_0^1 \int_0^1 k(t,s) f_1(s, x(s), y(s), x'(s), y'(s)) ds dB(t) \\
&\quad + \int_0^1 \int_0^1 k(t,s) f_2(s, x(s), y(s), x'(s), y'(s)) ds dA(t) < 0.
\end{aligned}
$$

Thus condition (H2) holds. Again taking $D = 30$, for any $a \in \mathbb{R}$, when $|a| > D$, we have

$$
\begin{aligned}
&a\kappa_1 \int_0^1 \int_0^1 k(t,s) f_1(s, \kappa_1 as, as, \kappa_1 a, a) ds dB(t) \\
&\quad + a \int_0^1 \int_0^1 k(t,s) f_2(s, \kappa_1 as, as, \kappa_1 a, a) ds dA(t) > 0.
\end{aligned}
$$

So condition (H3) holds. Hence from Theorem 3.1, BVP (1) has at least one solution $(x, y) \in C^1[0,1] \times C^1[0,1]$.

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