Supporting Information: Elastic deformations of spherical core-shell systems under an equatorial load

Jannis Kolker,*,† Lukas Fischer,*,‡ Andreas M. Menzel,‡ and Hartmut Löwen†

†*Institut für Theoretische Physik II: Weiche Materie, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany*

‡*Institut für Physik, Otto-von-Guericke-Universität Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany*

E-mail: jannis.kolker@hhu.de; lukas.fischer@ovgu.de

In this supporting information, the Navier-Cauchy equations and stress-strain relations Eqs. (1) and (2) in the main text, respectively, are presented in spherical coordinates for the problem under investigation. The further dependences of the coefficients $a_n^{(c)}$, $b_n^{(c)}$, $a_n^{(s)}$, $b_n^{(s)}$, $c_n^{(s)}$, $d_n^{(s)}$ on the dimensionless parameters $\frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}$, $\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}$, $\frac{R_{\mathrm{c}}}{R_{\mathrm{s}}}$, ν_{c}, ν_{s} and on the index n are listed in a two-step order. First, the dependence of the coefficients on the amplitude of the deformation $\frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}$, the ratio of the Young moduli $\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}$ and the ratio of the radii $\frac{R_{\mathrm{c}}}{R_{\mathrm{s}}}$ is shown and in the second step the dependence on the index n as well as on the Poisson ratios for core ν_{c} and shell ν_{s} is emphasised. At last the asymptotic behaviour for $n \to \infty$ is analysed for the Legendre polynomials and the general rescaled solutions for the radial component of the displacement field for the core and the shell.

Navier-Cauchy equations and stress-strain relations in spherical coordinates

Due to the special axial symmetry of the problem, the azimuthal component u_ϕ of the displacement field $\mathbf{u}(\mathbf{r})$ is zero and any ϕ-dependence vanishes. Therefore, the displacement field can be written as $\mathbf{u}(\mathbf{r}) = u_r(r,\theta)\mathbf{e}_r + u_\theta(r,\theta)\mathbf{e}_\theta$, where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ denote the radial and polar unit vectors, respectively. Then the homogeneous Navier-Cauchy equations, Eq. (1) in the main text, in spherical coordinates for the problem under investigation become

$$
\begin{aligned}
&2(1-\nu)\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r(r,\theta)\right)\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\theta(r,\theta)\right)\right)\right)\\
&\qquad -(1-2\nu)\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(\frac{\partial}{\partial r}\left(r u_\theta(r,\theta)\right)-\frac{\partial}{\partial\theta}u_r(r,\theta)\right)\right)\right)=0 \qquad (1)
\end{aligned}
$$

for the radial direction and

$$2(1-\nu)\left(\frac{1}{r^3}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}\left(r^2 u_r\left(r,\theta\right)\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta u_\theta\left(r,\theta\right)\right)\right)\right)\\ -(1-2\nu)\left(-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\left(ru_\theta\left(r,\theta\right)\right)-\frac{\partial}{\partial\theta}u_r\left(r,\theta\right)\right)\right)=0 \tag{2}$$

for the polar direction. The nontrivial components of the stress-strain relation, Eq. (2) in the main text, in spherical coordinates for the underlying problem read

$$\sigma_{rr}(r,\theta)=\frac{E}{1+\nu}\left(\varepsilon_{rr}(r,\theta)+\frac{\nu}{1-2\nu}\left(\varepsilon_{rr}(r,\theta)+\varepsilon_{\theta\theta}(r,\theta)\right)\right), \tag{3}$$

$$\sigma_{r\theta}(r,\theta)=\frac{E}{1+\nu}\varepsilon_{r\theta}(r,\theta), \tag{4}$$

$$\sigma_{\theta\theta}(r,\theta)=\frac{E}{1+\nu}\left(\varepsilon_{\theta\theta}(r,\theta)+\frac{\nu}{1-2\nu}\left(\varepsilon_{rr}(r,\theta)+\varepsilon_{\theta\theta}(r,\theta)\right)\right). \tag{5}$$

Here, in spherical coordinates, we inserted for the symmetric Cauchy stress tensor $\underline{\boldsymbol{\sigma}}\left(\mathbf{r}\right)=\sigma_{rr}(r,\theta)\mathbf{e}_r\otimes\mathbf{e}_r+\sigma_{r\theta}(r,\theta)\left(\mathbf{e}_\theta\otimes\mathbf{e}_r+\mathbf{e}_r\otimes\mathbf{e}_\theta\right)+\sigma_{\theta\theta}(r,\theta)\mathbf{e}_\theta\otimes\mathbf{e}_\theta$ and for the strain tensor $\underline{\boldsymbol{\varepsilon}}\left(\mathbf{r}\right)=\varepsilon_{rr}(r,\theta)\mathbf{e}_r\otimes\mathbf{e}_r+\varepsilon_{r\theta}(r,\theta)\left(\mathbf{e}_\theta\otimes\mathbf{e}_r+\mathbf{e}_r\otimes\mathbf{e}_\theta\right)+\varepsilon_{\theta\theta}(r,\theta)\mathbf{e}_\theta\otimes\mathbf{e}_\theta$, where $\otimes$ denotes the dyadic product.

Dependence of the coefficients $a_n^{(c)}, b_n^{(c)}, a_n^{(s)}, b_n^{(s)}, c_n^{(s)}, d_n^{(s)}$ on the amplitude of deformation $\frac{\lambda}{E_\mathbf{s}R_\mathbf{s}}$, the ratio of Young moduli $\frac{E_\mathbf{c}}{E_\mathbf{s}}$ and the ratio of radii $\frac{R_\mathbf{c}}{R_\mathbf{s}}$

The coefficients $a_n^{(c)}, b_n^{(c)}, a_n^{(s)}, b_n^{(s)}, c_n^{(s)}, d_n^{(s)}$ with $n \geq 0$ are listed and their dependence on the amplitude of deformation $\frac{\lambda}{E_\mathrm{s}R_\mathrm{s}}$, the ratio of Young moduli $\frac{E_\mathrm{c}}{E_\mathrm{s}}$ and the ratio of radii $\frac{R_\mathrm{c}}{R_\mathrm{s}}$ is highlighted. The expressions are found from the solutions of the relative deformation

$\frac{\mathbf{u}^{(c)}(R_{\mathrm{c}}\mathbf{e}_r)}{R_{\mathrm{c}}}$ and $\frac{\mathbf{u}^{(s)}(R_{\mathrm{s}}\mathbf{e}_r)}{R_{\mathrm{s}}}$, respectively:

$$\frac{a_n^{(c)}}{R_{\mathrm{c}}}R_{\mathrm{c}}^{n+1} = \frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}\frac{2n+1}{2}P_n(0)\left(\frac{R_{\mathrm{c}}}{R_{\mathrm{s}}}\right)^{(n-2)}\left[\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)\tilde{c}_{01,n}+\tilde{c}_{02,n}\right]\frac{1}{D},$$

$$\frac{b_n^{(c)}}{R_{\mathrm{c}}}R_{\mathrm{c}}^{n-1} = -\frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}\frac{2n+1}{2}P_n(0)\left(\frac{R_{\mathrm{c}}}{R_{\mathrm{s}}}\right)^{(n-2)}\left[\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)\tilde{c}_{03,n}+\tilde{c}_{04,n}\right]\frac{1}{D},$$

$$\begin{aligned}
\frac{a_n^{(s)}}{R_{\mathrm{s}}}R_{\mathrm{s}}^{n+1} &= \frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}\frac{2n+1}{2}P_n(0)\left[\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)^2\tilde{c}_{05,n}+\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)\tilde{c}_{06,n}+\tilde{c}_{07,n}\right]\frac{1}{D},\\
\frac{b_n^{(s)}}{R_{\mathrm{s}}}R_{\mathrm{s}}^{n-1} &= -\frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}\frac{2n+1}{2}P_n(0)\left[\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)^2\tilde{c}_{08,n}+\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)\tilde{c}_{09,n}+\tilde{c}_{10,n}\right]\frac{1}{D},\\
\frac{c_n^{(s)}}{R_{\mathrm{s}}}R_{\mathrm{s}}^{-n} &= \frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}\frac{2n+1}{2}P_n(0)\left(\frac{R_{\mathrm{c}}}{R_{\mathrm{s}}}\right)^{(2n-1)}\left[\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)^2\tilde{c}_{11,n}+\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)\tilde{c}_{12,n}+\tilde{c}_{13,n}\right]\frac{1}{D},\\
\frac{d_n^{(s)}}{R_{\mathrm{s}}}R_{\mathrm{s}}^{-(n+2)} &= -\frac{\lambda}{E_{\mathrm{s}}R_{\mathrm{s}}}\frac{2n+1}{2}P_n(0)\left(\frac{R_{\mathrm{c}}}{R_{\mathrm{s}}}\right)^{(2n+1)}\left[\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)^2\tilde{c}_{14,n}+\left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)\tilde{c}_{15,n}+\tilde{c}_{16,n}\right]\frac{1}{D},
\end{aligned}$$

where

$$D = \left(\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\right)^2\tilde{c}_{17,n}+\frac{E_{\mathrm{c}}}{E_{\mathrm{s}}}\tilde{c}_{18,n}+\tilde{c}_{19,n}. \tag{6}$$

The constants $\tilde{c}_{01,n}$ to $\tilde{c}_{19,n}$ are given below with their dependence on the ratio of radii $\frac{R_\mathrm{c}}{R_\mathrm{s}}$:

$$
\begin{aligned}
\tilde{c}_{01,n} =& c_{01,n} + c_{02,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{2} + c_{03,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)} + c_{04,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)}, \\
\tilde{c}_{02,n} =& c_{05,n} + c_{06,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{2} + c_{07,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)} + c_{08,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)}, \\
\tilde{c}_{03,n} =& c_{09,n} + c_{10,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{2} + c_{11,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{c}}\right)^{(2n+1)} + c_{12,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)}, \\
\tilde{c}_{04,n} =& c_{13,n} + c_{14,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{2} + c_{15,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)} + c_{16,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)}, \\
\tilde{c}_{05,n} =& c_{17,n} + c_{18,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n-1)} + c_{19,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)}, \\
\tilde{c}_{06,n} =& c_{20,n} + c_{21,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n-1)} + c_{22,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)}, \\
\tilde{c}_{07,n} =& c_{23,n} + c_{24,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n-1)} + c_{25,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)}, \\
\tilde{c}_{08,n} =& c_{26,n} + c_{27,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)} + c_{28,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)}, \\
\tilde{c}_{09,n} =& c_{29,n} + c_{30,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)} + c_{31,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)}, \\
\tilde{c}_{10,n} =& c_{32,n} + c_{33,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+1)} + c_{34,n}\left(\frac{R_\mathrm{c}}{R_\mathrm{s}}\right)^{(2n+3)},
\end{aligned}
$$

$$\tilde{c}_{11,n} = c_{35,n} + c_{36,n}\left(\frac{R_c}{R_s}\right)^2 + c_{37,n}\left(\frac{R_c}{R_s}\right)^{(2n+3)},$$
$$\tilde{c}_{12,n} = c_{38,n} + c_{39,n}\left(\frac{R_c}{R_s}\right)^2 + c_{40,n}\left(\frac{R_c}{R_s}\right)^{(2n+3)},$$
$$\tilde{c}_{13,n} = c_{41,n} + c_{42,n}\left(\frac{R_c}{R_s}\right)^2 + c_{43,n}\left(\frac{R_c}{R_s}\right)^{(2n+3)},$$
$$\tilde{c}_{14,n} = c_{44,n} + c_{45,n}\left(\frac{R_c}{R_s}\right)^2 + c_{46,n}\left(\frac{R_c}{R_s}\right)^{(2n+1)},$$
$$\tilde{c}_{15,n} = c_{47,n} + c_{48,n}\left(\frac{R_c}{R_s}\right)^2 + c_{49,n}\left(\frac{R_c}{R_s}\right)^{(2n+1)},$$
$$\tilde{c}_{16,n} = c_{50,n} + c_{51,n}\left(\frac{R_c}{R_s}\right)^2 + c_{52,n}\left(\frac{R_c}{R_s}\right)^{(2n+1)},$$
$$\tilde{c}_{17,n} = c_{53,n} + c_{54,n}\left(\frac{R_c}{R_s}\right)^{(2n-1)} + c_{55,n}\left(\frac{R_c}{R_s}\right)^{(2n+1)} + c_{56,n}\left(\frac{R_c}{R_s}\right)^{(2n+3)} + c_{57,n}\left(\frac{R_c}{R_s}\right)^{(4n+2)},$$
$$\tilde{c}_{18,n} = c_{58,n} + c_{59,n}\left(\frac{R_c}{R_s}\right)^{(2n-1)} + c_{60,n}\left(\frac{R_c}{R_s}\right)^{(2n+1)} + c_{61,n}\left(\frac{R_c}{R_s}\right)^{(2n+3)} + c_{62,n}\left(\frac{R_c}{R_s}\right)^{(4n+2)},$$
$$\tilde{c}_{19,n} = c_{63,n} + c_{64,n}\left(\frac{R_c}{R_s}\right)^{(2n-1)} + c_{65,n}\left(\frac{R_c}{R_s}\right)^{(2n+1)} + c_{66,n}\left(\frac{R_c}{R_s}\right)^{(2n+3)} + c_{67,n}\left(\frac{R_c}{R_s}\right)^{(4n+2)}.$$

Dependence of the constants $c_{01,n}$ to $c_{67,n}$ on the index n, the Poisson ratio of the core ν_c and of the shell ν_s

The constants $c_{01,n}$ to $c_{67,n}$ only depend on the index n, the Poisson ratio of the core ν_c and of the shell ν_s. They are listed below:

$$c_{01,n} = 0,$$
$$c_{02,n} = -\frac{4(-1+n)^2(3+8n+4n^2)(-1+\nu_s)(-2-3n+2\nu_s+4n\nu_s)}{(1+\nu_c)(1+\nu_s)^2},$$
$$c_{03,n} = -\frac{2(1+2n)^2(-3+n+2n^2)(-1+\nu_s)(-2+n^2+2\nu_s)}{(1+\nu_c)(1+\nu_s)^2},$$
$$c_{04,n} = \frac{2n(2+n)(3-n-14n^2+4n^3+8n^4)(-1+\nu_s)}{(1+\nu_c)(1+\nu_s)^2},$$
$$c_{05,n} = 0,$$
$$c_{06,n} = \frac{4(-1+n)(3+8n+4n^2)(-1+\nu_s)(1+n+n^2-\nu_s-2n\nu_s)}{(1+\nu_s)^3},$$

$$c_{07,n} = \frac{2(1+2n)^2(-3+n+2n^2)(-1+\nu_\mathrm{s})(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{08,n} = -\frac{2n(2+n)(3-n-14n^2+4n^3+8n^4)(-1+\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{09,n} = \frac{4(-1+4n^2)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-1+\nu_\mathrm{s})(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{10,n} = -\frac{4(-1+n)(3+11n+12n^2+4n^3)(-1+\nu_\mathrm{s})(5-\nu_\mathrm{c}-6\nu_\mathrm{s}+2n(-1+\nu_\mathrm{c}+\nu_\mathrm{s})+n^2(-2+4\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{11,n} = -\frac{2(3+2n)^2(-1-2n+n^2+2n^3)(-1+\nu_\mathrm{s})(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{12,n} = 2(2+n)(-1+4n^2)(-1+\nu_\mathrm{s})\left[\frac{5n^3+2n^4+n^2(6-8\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2} + \frac{-4(1+\nu_\mathrm{c})(-1+2\nu_\mathrm{s})-n(1+8\nu_\mathrm{s}+4\nu_\mathrm{c}(-3+4\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right],$$

$$c_{13,n} = -\frac{4(-2-n+8n^2+4n^3)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-1+\nu_\mathrm{s})(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{14,n} = \frac{4(-1+n)(3+11n+12n^2+4n^3)(-1+\nu_\mathrm{s})(5-4\nu_\mathrm{c}+n^2(-2+4\nu_\mathrm{c})-3\nu_\mathrm{s}+n(6\nu_\mathrm{c}-2(1+\nu_\mathrm{s})))}{(1+\nu_\mathrm{s})^3},$$

$$c_{15,n} = \frac{2(3+2n)^2(-1-2n+n^2+2n^3)(-1+\nu_\mathrm{s})(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{16,n} = -2(2+n)(-1+4n^2)(-1+\nu_\mathrm{s})\left[\frac{5n^3+2n^4+n^2(6-8\nu_\mathrm{c})}{(1+\nu_\mathrm{s})^3} + \frac{-4(-1+2\nu_\mathrm{c})(1+\nu_\mathrm{s})-n(1-12\nu_\mathrm{s}+8\nu_\mathrm{c}(1+2\nu_\mathrm{s}))}{(1+\nu_\mathrm{s})^3}\right],$$

$$c_{17,n} = \frac{4(-1+n)^2(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-2-3n+2\nu_\mathrm{s}+4n\nu_\mathrm{s})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{18,n} = \frac{2(-1+n)(1+2n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{19,n} = -\frac{2(-1+n)n(2+n)(-1+2n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{20,n} = -4(-1+n)\left[\frac{-3(-1+3\nu_\mathrm{c})(-1+\nu_\mathrm{s})+n^2(-4+\nu_\mathrm{c}(9-16\nu_\mathrm{s})+9\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2} + \frac{n(-14+\nu_\mathrm{c}(27-32\nu_\mathrm{s})+15\nu_\mathrm{s})+4n^3(5-6\nu_\mathrm{s}+\nu_\mathrm{c}(-6+8\nu_\mathrm{s}))+2n^4(5-6\nu_\mathrm{s}+\nu_\mathrm{c}(-6+8\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right],$$

$$c_{21,n} = -\frac{2(-1-n+2n^2)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{22,n} = \frac{2n(2+n)(1-3n+2n^2)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{23,n} = \frac{4(-1+n)(2+n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(1+n+n^2-\nu_\mathrm{s}-2n\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{24,n} = \frac{2(-2-3n+3n^2+2n^3)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{25,n} = -\frac{2n(2+n)^2(1-3n+2n^2)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))}{(1+\nu_\mathrm{s})^3},$$

$$c_{26,n} = \frac{4(-1+n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-2-3n+2\nu_\mathrm{s}+4n\nu_\mathrm{s})(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})(1+\nu_\mathrm{c})^2},$$

$$c_{27,n} = \frac{2(-1+n)(3+5n+2n^2)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{28,n} = -\frac{2(-1+n)(2+n)(1+2n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(8+n+n^2-24\nu_\mathrm{s}+16\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{29,n} = -4(-1+2n+n^2+2\nu_\mathrm{s})\left[\frac{-3(-1+3\nu_\mathrm{c})(-1+\nu_\mathrm{s})+n^2(-4+\nu_\mathrm{c}(9-16\nu_\mathrm{s})+9\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2} + \frac{n(-14+\nu_\mathrm{c}(27-32\nu_\mathrm{s})+15\nu_\mathrm{s})+4n^3(5-6\nu_\mathrm{s}+\nu_\mathrm{c}(-6+8\nu_\mathrm{s}))+2n^4(5-6\nu_\mathrm{s}+\nu_\mathrm{c}(-6+8\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right],$$

$$c_{30,n} = -\frac{2(-3-2n+3n^2+2n^3)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{31,n} = 2(2+n)(1+2n)\left[\frac{6n^4(-1+2\nu_\mathrm{c})+n^5(-2+4\nu_\mathrm{c})-12(-1+\nu_\mathrm{s})(-1+\nu_\mathrm{c}+2\nu_\mathrm{c}\nu_\mathrm{s})-n(11+\nu_\mathrm{c}-4\nu_\mathrm{s}-28\nu_\mathrm{c}\nu_\mathrm{s}-8\nu_\mathrm{s}^2+32\nu_\mathrm{c}\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right.$$
$$\left.+\frac{n^3(9-8\nu_\mathrm{s}+\nu_\mathrm{c}(-15+16\nu_\mathrm{s}))+2n^2(3+16\nu_\mathrm{s}-16\nu_\mathrm{s}^2+2\nu_\mathrm{c}(-7-4\nu_\mathrm{s}+8\nu_\mathrm{s}^2))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right],$$

$$c_{32,n} = \frac{4(2+n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-1+2n+n^2+2\nu_\mathrm{s})(1+n+n^2-\nu_\mathrm{s}-2n\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{33,n} = \frac{2(2+n)(-3-2n+3n^2+2n^3)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-2+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{34,n} = -\frac{2(2+n)(1+2n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(4-2n-n^2+2n^3+n^4-4\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{s})^3},$$

$$c_{35,n} = -\frac{2(-1+n)(1+2n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})(1+\nu_\mathrm{c})^2},$$

$$c_{36,n} = \frac{2(-1+n)^2(3+5n+2n^2)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{37,n} = \frac{4(-1+n)(2+n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-1+2\nu_\mathrm{s}+n(-3+4\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{38,n} = \frac{2(-1-n+2n^2)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{39,n} = -\frac{2(-1+n)^2(3+5n+2n^2)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{40,n} = -\frac{4(-1+n)(2+n)(-2+\nu_\mathrm{c}+\nu_\mathrm{s}+4\nu_\mathrm{c}\nu_\mathrm{s}+n^3(-6+4\nu_\mathrm{c}+4\nu_\mathrm{s})+8n^2(-1+2\nu_\mathrm{c}\nu_\mathrm{s})+n(-8+\nu_\mathrm{c}+\nu_\mathrm{s}+16\nu_\mathrm{c}\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{41,n} = -\frac{2(-2-3n+3n^2+2n^3)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{42,n} = \frac{2(-1+n)^2(2+n)(3+5n+2n^2)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))}{(1+\nu_\mathrm{s})^3},$$

$$c_{43,n} = \frac{4(-1+n)(2+n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(1+n+n^2+\nu_\mathrm{s}+2n\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{44,n} = \frac{2(-1+n)n(-1+2n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})(1+\nu_\mathrm{c})^2},$$

$$c_{45,n} = -\frac{2(-1+n)^2(1+2n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(8+n+n^2-24\nu_\mathrm{s}+16\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{46,n} = -\frac{4(-1+n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(-2+n^2+2\nu_\mathrm{s})(-1+2\nu_\mathrm{s}+n(-3+4\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})},$$

$$c_{47,n} = -\frac{2n(1-3n+2n^2)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{48,n} = 2(-1+n)(1+2n)\left[\frac{6n^4(-1+2\nu_\mathrm{c})+n^5(-2+4\nu_\mathrm{c})-12(-1+\nu_\mathrm{s})(-1+\nu_\mathrm{c}+2\nu_\mathrm{c}\nu_\mathrm{s})-n(11+\nu_\mathrm{c}-4\nu_\mathrm{s}-28\nu_\mathrm{c}\nu_\mathrm{s}-8\nu_\mathrm{s}^2+32\nu_\mathrm{c}\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right.$$
$$\left.+\frac{n^3(9-8\nu_\mathrm{s}+\nu_\mathrm{c}(-15+16\nu_\mathrm{s}))+2n^2(3+16\nu_\mathrm{s}-16\nu_\mathrm{s}^2+2\nu_\mathrm{c}(-7-4\nu_\mathrm{s}+8\nu_\mathrm{s}^2))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2}\right],$$

$$c_{49,n} = \frac{4(-1+n)(-2+n^2+2\nu_\mathrm{s})(-2+\nu_\mathrm{c}+\nu_\mathrm{s}+4\nu_\mathrm{c}\nu_\mathrm{s}+n^3(-6+4\nu_\mathrm{c}+4\nu_\mathrm{s})+8n^2(-1+2\nu_\mathrm{c}\nu_\mathrm{s})+n(-8+\nu_\mathrm{c}+\nu_\mathrm{s}+16\nu_\mathrm{c}\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^2},$$

$$c_{50,n} = \frac{2n(2-5n+n^2+2n^3)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-1+2n+n^2+2\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{51,n} = -\frac{2(-1+n)(1+2n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(4-2n-n^2+2n^3+n^4-4\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{s})^3},$$

$$c_{52,n} = -\frac{4(-1+n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(-2+n^2+2\nu_\mathrm{s})(1+n+n^2+\nu_\mathrm{s}+2n\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^3},$$

$$c_{53,n} = -\frac{8(-1+n)^2(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(1+n+n^2+\nu_\mathrm{s}+2n\nu_\mathrm{s})(-2-3n+2\nu_\mathrm{s}+4n\nu_\mathrm{s})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})^2},$$

$$c_{54,n} = \frac{2(-1+n)(1+2n)^2(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(4-2n-n^2+2n^3+n^4-4\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})^2},$$

$$c_{55,n} = -\frac{4(-1+n)^2n(-6-n+17n^2+16n^3+4n^4)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})^2},$$

$$c_{56,n} = \frac{2(-1+n)^2(2+n)(1+2n)^2(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(8+n+n^2-24\nu_\mathrm{s}+16\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})^2},$$

$$c_{57,n} = -\frac{8(-1+n)(2+n)(1+n+n^2+\nu_\mathrm{c}+2n\nu_\mathrm{c})(1+n+n^2-\nu_\mathrm{s}-2n\nu_\mathrm{s})(-1+2\nu_\mathrm{s}+n(-3+4\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})^2(1+\nu_\mathrm{s})^2},$$

$$c_{58,n} = 8(-1+n)(1+n+n^2+\nu_\mathrm{s}+2n\nu_\mathrm{s})\left[\frac{-3(-1+3\nu_\mathrm{c})(-1+\nu_\mathrm{s})+n^2(-4+\nu_\mathrm{c}(9-16\nu_\mathrm{s})+9\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3}\right.$$
$$\left.+\frac{n(-14+\nu_\mathrm{c}(27-32\nu_\mathrm{s})+15\nu_\mathrm{s})+4n^3(5-6\nu_\mathrm{s}+\nu_\mathrm{c}(-6+8\nu_\mathrm{s}))+2n^4(5-6\nu_\mathrm{s}+\nu_\mathrm{c}(-6+8\nu_\mathrm{s}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3}\right],$$

$$c_{59,n} = -\frac{2(-1+n)(1+2n)^2(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))(4-2n-n^2+2n^3+n^4-4\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3},$$

$$c_{60,n} = \frac{4(-1+n)^2n(-6-n+17n^2+16n^3+4n^4)(-1+5\nu_\mathrm{c}+6n(-1+2\nu_\mathrm{c})+n^2(-2+4\nu_\mathrm{c}))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3},$$

$$c_{61,n} = -2(1+2n)^2(-2+n+n^2)\left[\frac{6n^4(-1+2\nu_\mathrm{c})+n^5(-2+4\nu_\mathrm{c})-12(-1+\nu_\mathrm{s})(-1+\nu_\mathrm{c}+2\nu_\mathrm{c}\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3}\right.$$
$$\left.+\frac{-n(11+\nu_\mathrm{c}-4\nu_\mathrm{s}-28\nu_\mathrm{c}\nu_\mathrm{s}-8\nu_\mathrm{s}^2+32\nu_\mathrm{c}\nu_\mathrm{s}^2)+n^3(9-8\nu_\mathrm{s}+\nu_\mathrm{c}(-15+16\nu_\mathrm{s}))+2n^2(3+16\nu_\mathrm{s}-16\nu_\mathrm{s}^2+2\nu_\mathrm{c}(-7-4\nu_\mathrm{s}+8\nu_\mathrm{s}^2))}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3}\right],$$

$$c_{62,n} = 8(-1+n)(2+n)(1+n+n^2-\nu_\mathrm{s}-2n\nu_\mathrm{s})\left[\frac{-2+\nu_\mathrm{c}+\nu_\mathrm{s}+4\nu_\mathrm{c}\nu_\mathrm{s}+n^3(-6+4\nu_\mathrm{c}+4\nu_\mathrm{s})+8n^2(-1+2\nu_\mathrm{c}\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3}\right.$$
$$\left.+\frac{n(-8+\nu_\mathrm{c}+\nu_\mathrm{s}+16\nu_\mathrm{c}\nu_\mathrm{s})}{(1+\nu_\mathrm{c})(1+\nu_\mathrm{s})^3}\right],$$

$$c_{63,n} = -\frac{8(-1+n)(2+n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(1+n+n^2-\nu_\mathrm{s}-2n\nu_\mathrm{s})(1+n+n^2+\nu_\mathrm{s}+2n\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^4},$$

$$c_{64,n} = \frac{2(1+2n)^2(-2+n+n^2)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(4-2n-n^2+2n^3+n^4-4\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{s})^4},$$

$$c_{65,n} = -\frac{4n(-2+n+n^2)^2(-3+n+8n^2+4n^3)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))}{(1+\nu_\mathrm{s})^4},$$

$$c_{66,n} = \frac{2(-1+n)(2+n)(1+2n)^2(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(4-2n-n^2+2n^3+n^4-4\nu_\mathrm{s}^2)}{(1+\nu_\mathrm{s})^4},$$

$$c_{67,n} = -\frac{8(-1+n)(2+n)(-1+2\nu_\mathrm{c}+n(-3+4\nu_\mathrm{c}))(1+n+n^2-\nu_\mathrm{s}-2n\nu_\mathrm{s})(1+n+n^2+\nu_\mathrm{s}+2n\nu_\mathrm{s})}{(1+\nu_\mathrm{s})^4}.$$

Asymptotic behaviour of the Legendre polynomials P_n and the general rescaled solutions for the radial component of the displacement field for the core $u_r^{\mathbf{(c)}}/R_c$ and for the shell $u_r^{\mathbf{(s)}}/R_s$

For the Legendre polynomials $P_n(\cos\theta)$ with $\theta = \pi/2$, the dependence on the index n is as follows[1]

$$P_n(0) = \begin{cases} \frac{(-1)^m}{2^{2m}} \frac{(2m)!}{(m!)^2} & \text{for} \quad n = 2m, \\ 0 & \text{for} \quad n = 2m+1. \end{cases} \tag{7}$$

Let $a_m = \frac{1}{2^{2m}} \frac{(2m)!}{(m!)^2}$. To calculate the asymptotic behaviour of this coefficient for $m \to \infty$, Stirling's formula can be used:

$$N! = \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right), \tag{8}$$

where e denotes Euler's number. Applying this formula to a_m leads (for large m) to:

$$\begin{aligned} a_m &\approx \frac{1}{2^{2m}} \frac{\sqrt{2\pi 2m}}{2\pi m} \left(\frac{2m}{e}\right)^{2m} \left(\frac{e}{m}\right)^{2m} \\ &= \frac{2^{2m}}{2^{2m}} \frac{\sqrt{2\pi 2m}}{2\pi m} \\ &= \frac{1}{\sqrt{\pi m}}. \end{aligned} \tag{9}$$

Thus, the following asymptotic behaviour for $P_n(0)$ results:

$$P_n(0) \approx \begin{cases} \sqrt{\frac{2}{\pi n}} & \text{for} \quad n \,\text{even}, \\ 0 & \text{for} \quad n \,\text{odd}. \end{cases} \tag{10}$$

Furthermore, the dependence on the index n for the angles $\theta = 0, \pi$ gives[1]

$$P_n\left(1\right) = 1, \tag{11}$$

$$P_n\left(-1\right) = \begin{cases} 1 & \text{for} \quad n\,\text{even}, \\ -1 & \text{for} \quad n\,\text{odd}. \end{cases} \tag{12}$$

The case of n being odd is, due to the assumed mirror symmetry, irrelevant for the investigated problem, therefore $P_n(\cos 0) = P_n(\cos \pi) = 1$ holds true.

The general rescaled solution for the radial component of the displacement field for the core $u_r^{(\mathrm{c})}/R_c$ is obtained at the core radius R_c as follows

$$\frac{u_r^{(\mathrm{c})}(R_c\mathbf{e}_r)}{R_c} = \sum_{n=0}^{\infty} G_{r,n}^{(\mathrm{c})}\left(\frac{\lambda}{E_sR_s}, \frac{E_c}{E_s}, \frac{R_c}{R_s}, \nu_c, \nu_s\right) \frac{2n+1}{2} P_n\left(0\right) P_n\left(\cos\theta\right) \tag{13}$$

where $G_{r,n}^{(\mathrm{c})}\left(\lambda/(E_sR_s), E_c/E_s, R_c/R_s, \nu_c, \nu_s\right)$ is the corresponding kernel function of the core and the remaining factors in the sum result from the expansion of the Dirac delta function in Legendre polynomials. In terms of the coefficients $a_n^{(c)}$ and $b_n^{(c)}$, (13) can also be written as

$$\begin{aligned} \frac{u_r^{(\mathrm{c})}\left(R_c\mathbf{e}_r\right)}{R_c} &= \sum_{n=0}^{\infty}\left(\frac{a_n^{(c)}}{R_c}R_c^{n+1}(n+1)(-2+n+4\nu_c) + \frac{b_n^{(c)}}{R_c}R_c^{n-1}n\right)P_n\left(\cos\theta\right) \\ &= \sum_{n=0}^{\infty}\frac{2n+1}{2}P_n(0)P_n\left(\cos\theta\right)\left(\frac{R_c}{R_s}\right)^{(n-2)}\frac{\lambda}{E_sR_s} \\ &\qquad \times \frac{1}{D}\left(\underbrace{\left[\left(\frac{E_c}{E_s}\right)\tilde{c}_{01,n} + \tilde{c}_{02,n}\right](n+1)(-2+n+4\nu_c)}_{I} - \underbrace{\left[\left(\frac{E_c}{E_s}\right)\tilde{c}_{03,n} + \tilde{c}_{04,n}\right]n}_{II}\right). \end{aligned} \tag{14}$$

Comparing the solution for $u_r^{(\mathrm{c})}/R_c$ here with that in (13), it can be concluded that the kernel function of the core $G_{r,n}^{(\mathrm{c})}$ is the product of the factors $(R_c/R_s)^{(n-2)}$, $\lambda/(E_sR_s)$, $1/D$

(see Eq. (6)) and the sum of 14I + 14II. By multiplying the sum 14I + 14II by $1/D$, an order in index n of $\mathcal{O}(1)$ can be proved in the asymptotic behaviour of the limit $n \to \infty$ for $R_c/R_s < 1$. Therefore, the factor $(R_c/R_s)^{(n-2)}$ is the dominant factor in the asymptotic behaviour for the limit $n \to \infty$ of the kernel function of the core $G^{(\mathrm{c})}_{r,n}$. Combined with the n-dependence of the Legendre polynomials $P_n(\cos\theta)$ the general rescaled radial solution of the core $u^{(\mathrm{c})}_r/R_c$ at the core radius R_c gives a convergent series at the poles and at the equator, due to the $(R_c/R_s)^n$-dependence ($R_c/R_s < 1$, exponential decrease).

The general rescaled solution for the radial component of the displacement field for the shell $u^{(\mathrm{s})}_r/R_s$ is obtained at the outer shell radius R_s as

$$\frac{u^{(\mathrm{s})}_r(R_s\mathbf{e}_r)}{R_s} = \sum_{n=0}^{\infty} G^{(\mathrm{s})}_{r,n}\left(\frac{\lambda}{E_sR_s}, \frac{E_c}{E_s}, \frac{R_c}{R_s}, \nu_c, \nu_s\right)\frac{2n+1}{2}P_n(0)\,P_n(\cos\theta) \tag{15}$$

where $G^{(\mathrm{s})}_{r,n}(\lambda/(E_sR_s), E_c/E_s, R_c/R_s, \nu_c, \nu_s)$ is the corresponding kernel function of the shell and the remaining factors are the same as for the core solution. In terms of the coefficients

$a_n^{(s)}$, $b_n^{(s)}$, $c_n^{(c)}$ and $d_n^{(c)}$, (15) can also be written as

$$\begin{aligned}
\frac{u_r^{(\mathrm{s})}\left(R_s\mathbf{e}_r\right)}{R_s} &= \sum_{n=0}^{\infty}\left(\frac{a_n^{(s)}}{R_s}R_s^{n+1}(n+1)(-2+n+4\nu_s)+\frac{b_n^{(s)}}{R_s}R_s^{n-1}n\right.\\
&\qquad\left.+\frac{c_n^{(s)}}{R_s}R_s^{-n}n(3+n-4\nu_s)-\frac{d_n^{(s)}}{R_s}R_2^{-(n+2)}(n+1)\right)P_n\left(\cos\theta\right)\\
&= \sum_{n=0}^{\infty}\frac{2n+1}{2}P_n(0)P_n\left(\cos\theta\right)\frac{\lambda}{E_sR_s}\\
&\qquad\times\frac{1}{D}\left(\underbrace{\left[\left(\frac{E_c}{E_s}\right)^2\tilde{c}_{05,n}+\left(\frac{E_c}{E_s}\right)\tilde{c}_{06,n}+\tilde{c}_{07,n}\right](n+1)(-2+n+4\nu_s)}_{I}\right.\\
&\qquad\underbrace{-\left[\left(\frac{E_c}{E_s}\right)^2\tilde{c}_{08,n}+\left(\frac{E_c}{E_s}\right)\tilde{c}_{09,n}+\tilde{c}_{10,n}\right]n}_{II}\\
&\qquad\underbrace{+\left(\frac{R_c}{R_s}\right)^{(2n-1)}\left[\left(\frac{E_c}{E_s}\right)^2\tilde{c}_{11,n}+\left(\frac{E_c}{E_s}\right)\tilde{c}_{12,n}+\tilde{c}_{13,n}\right]n(3+n-4\nu_s)}_{III}\\
&\qquad\left.\underbrace{+\left(\frac{R_c}{R_s}\right)^{(2n+1)}\left[\left(\frac{E_c}{E_s}\right)^2\tilde{c}_{14,n}+\left(\frac{E_c}{E_s}\right)\tilde{c}_{15,n}+\tilde{c}_{16,n}\right](n+1)}_{IV}\right)
\end{aligned}\tag{16}$$

By comparing the solution for $u_r^{(\mathrm{s})}/R_s$ with that in (15), it can be concluded that the kernel function of the shell $G_{r,n}^{(\mathrm{s})}$ is the product of the factors $\lambda/(E_sR_s)$, $1/D$ and the sum of 16I + 16II + 16III + 16IV. By multiplying the sum 16I + 16II by $1/D$, an order in index n of $\mathcal{O}(1/n)$ can be proved in the asymptotic behaviour of the limit $n\to\infty$ for $R_c/R_s<1$. Multiplying the sum 16III + 16IV by $1/D$ leads to a dominant factor of $(R_c/R_s)^{2n}$ under the same conditions. Therefore, the asymptotic behaviour for $n\to\infty$ is proportional to $1/n$ for the kernel function of the shell $G_{r,n}^{(\mathrm{s})}$. Combined with the n-dependence of the Legendre polynomials $P_n\left(\cos\theta\right)$ the general rescaled radial solution for the shell $u_r^{(\mathrm{s})}/R_s$ at the outer shell radius R_s results in a divergent series at the equator ($\theta = \pi/2$), due to the $1/n$-dependence of $G^{(s)}_{r,n}$ (harmonic series) and a convergent series at the poles ($\theta = 0, \pi$), due to the property of the Legendre polynomials at the poles (alternating series and a monotonic decrease to zero of the absolute value of the summands).

References

(1) Arfken, G., Weber H.; *Mathematical Methods for Physicists*; Elsevier Academic Press, United Kingdom, 2005.