Queueing Systems manuscript No.
(will be inserted by the editor)

Online supplement: The analysis of batch sojourn-times in polling systems

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Received: date / Accepted: date

A Locally-gated service

In this section, we study batch sojourn-times in a polling system with locally-gated service. In Sect. A.1 and Sect. A.2 we will study the joint queue-length distribution and the LST of the batch sojourn-time distribution. Instead of providing a thorough analysis, we present the differences with the analysis of Sect. 4. Finally, in Sect. A.3 a Mean Value Analysis (MVA) is presented to calculate the mean batch sojourn-time.

A.1 The joint queue-length distributions

Similar as in Sect. 4.1, we start by defining the laws of motions in case of locally-gated service. For this we distinguish between customers that are standing behind of the gate and those who are standing before the gate [1]. Customers that are standing behind the gate will be served in the current cycle, whereas customers before the gate will only be served in the next cycle. Let $\widetilde{LB}^{(V_i)}(\boldsymbol{z})$, $\widetilde{LB}^{(S_i)}(\boldsymbol{z})$, $\widetilde{LC}^{(S_i)}(\boldsymbol{z})$, and $\widetilde{LC}^{(V_i)}(\boldsymbol{z})$ be the joint queue-length PGF at *visit*/*switch-over* beginnings and completions at Q_i, for $i = 1, \dots, N$, where $\boldsymbol{z} = (z_1, \dots, z_N, z_G)$ is an $N+1$ dimensional vector. The first N elements correspond with the number of customers that are standing behind gate Q_i, $i = 1, \dots, N$, whereas element $N+1$, z_G, is used during visit periods to correspond with the number of customers that are currently standing before the gate at the queue that is currently being visited.

Then the law of motions for locally-gated service are as follows,

$$\begin{aligned}\widetilde{LC}^{(V_i)}(\boldsymbol{z}) =&\widetilde{LB}^{(V_i)}\Big(z_1, \dots, z_{i-1}, \widetilde{B}_i\left(\lambda - \lambda K\left(z_1, \dots, z_{i-1}, z_G, z_{i+1}, \dots, z_N\right)\right), \\ & z_{i+1}, \dots, z_N, z_G\Big), \qquad \text{(A.1)}\\ \widetilde{LB}^{(S_i)}(\boldsymbol{z}) =&\widetilde{LC}^{(V_i)}\left(z_1, \dots, z_N, z_i\right), \qquad \text{(A.2)}\end{aligned}$$

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$$\widetilde{LC}^{(S_i)}(\boldsymbol{z}) = \widetilde{LB}^{(S_i)}(\boldsymbol{z})\,\widetilde{S}_i\left(\lambda - \lambda\widetilde{K}(z_1,\ldots,z_{i-1},z_i,z_{i+1},\ldots,z_N)\right), \tag{A.3}$$

$$\widetilde{LB}^{(V_{i+1})}(\boldsymbol{z}) = \widetilde{LC}^{(S_i)}(\boldsymbol{z}), \tag{A.4}$$

Equation (A.1) states that the queue-length in Q_j, $j \neq i$ at the end of visit period V_i is composed of the number of customers already at Q_j at the visit beginning plus all the customers that arrived in the system during the current visit period. However for Q_i, only the customers that were standing behind the gate are served before the end of the visit completion; customers that arrived to Q_i during this visit period are placed before the gate and will be served during the next visit to Q_i. In (A.2) it can be seen that the PGF of a visit completion corresponds to the PGF of the next switch-over beginning, except that the customer standing before the gate in Q_i are now placed behind the gate. Finally, the interpretation of (A.3) and (A.4) is the same as for (4) and (5).

In order to define the PGF of the joint queue-length distribution, Eisenberg's relationship (7) is also valid for locally-gated service. However, the joint queue-length distribution at service beginnings and completions (8) should be modified to,

$$\widetilde{LC}^{(B_i)}(\boldsymbol{z}) = \widetilde{LB}^{(B_i)}(\boldsymbol{z}) \times \left[\widetilde{B}_i\left(\lambda - \lambda\widetilde{K}(z_1,\ldots,z_{i-1},z_G,z_{i+1},\ldots,z_N)\right)/z_i\right], \tag{A.5}$$

since during a service period in Q_i arriving customers who join Q_i are placed before the gate. A similar modification also applies for the PGF of the joint queue-length distributions at an arbitrary moment during V_i,

$$\tilde{L}^{(V_i)}(\boldsymbol{z}) = \widetilde{LB}^{(B_i)}(\boldsymbol{z})\,\frac{1 - \widetilde{B}_i\left(\lambda - \lambda\widetilde{K}(z_1,\ldots,z_{i-1},z_G,z_{i+1},\ldots,z_N)\right)}{E(B_i)\left(\lambda - \lambda\widetilde{K}(z_1,\ldots,z_{i-1},z_G,z_{i+1},\ldots,z_N)\right)}. \tag{A.6}$$

Then, all the other results from Sect. 4.1 can be easily modified for locally-gated service.

A.2 Batch sojourn-time distribution

In the following section we derive the LST of the steady-state batch sojourn-time distribution for locally-gated service. Assume than an arriving customer batch $\boldsymbol{k}$ enters the system while the server is currently within visit period V_{j-1} or switch-over period S_{j-1} such that the last customer in the batch will be served in Q_i. This means $k_i > 0$ and all the other customers arriving in the same batch should be served before the next visit to Q_i; $k_l \geq 0$, $l = j,\ldots,i-1$, and $k_l = 0$ elsewhere. Whenever a customer arrives in the same queue that is currently being visited, then this customer will be placed before the gate. As a consequence, this customer will be served last in the batch since the server will visit first all the other queues before serving this customer.

Similar as for exhaustive service, let $B_{j,i}$ $i,j = 1,\ldots,N$, be the service of a tagged customer in Q_j plus all its decedents that will be served before or during the next visit to Q_i. Since during a service period in Q_j incoming customers to Q_j are placed before the gate, we have

$$B_{j,i} = \begin{cases} B_j & \text{if } i = j, \\ B_j + \sum_{l=j+1}^{i}{}' \sum_{m=1}^{N_l(B_j)} B_{l_m,i}, & \text{otherwise,} \end{cases} \tag{A.7}$$

where B_j is the service time of the tagged customer in Q_j, $N_l(B_j)$ denotes the number of customers that arrive in Q_l during the service time of the tagged customer in Q_j, and $B_{l_m,i}$ is a sequence of (independent) of $B_{l,i}$'s. Let $\widetilde{B}_{j,i}(.)$ be the LST which is given by,

$$\widetilde{B}_{j,i}(\omega) = \widetilde{B}_j\left(\omega + \lambda(1 - \widetilde{K}(\boldsymbol{B_{j+1,i}}))\right), \tag{A.8}$$

where $\boldsymbol{B_{j+1,i}}$ is an N-dimensional vector similar defined as (16). We define $\boldsymbol{B^*_{j,i}}$ as an $N+1$-dimensional vector defined as follows,

$$(\boldsymbol{B^*_{j,i}})_l = \begin{cases} \widetilde{B}_i(\omega), & \text{if } l = i, \\ 1, & \text{if } l = N+1, \\ (\boldsymbol{B_{j,i-1}})_l, & \text{otherwise.} \end{cases} \tag{A.9}$$

Finally, let $\boldsymbol{B^G_{j,i}}$, $i, j = 1, \dots, N$, be an $N+1$-dimensional vector defined as for $j \neq i$,

$$(\boldsymbol{B^G_{j,i}})_l = \begin{cases} (\boldsymbol{B_{j,i}})_l & \text{if } l = j, \dots, i, \\ 1, & \text{otherwise,} \end{cases} \tag{A.10}$$

and for $j = i$,

$$\begin{aligned} \boldsymbol{B^G_{i,i}} = \Big(&\widetilde{B}_{1,i-1}(\omega), \dots, \widetilde{B}_{i-1,i-1}(\omega) \\ &\widetilde{B}_i\left(\omega + \lambda(1 - \widetilde{K}(\widetilde{B}_{1,i-1}(\omega), \dots, \widetilde{B}_i(\omega), \dots, \widetilde{B}_{N,i-1}(\omega)))\right), \\ &\widetilde{B}_{i+1,i-1}(\omega) \dots, \widetilde{B}_{N,i-1}(\omega), \widetilde{B}_i(\omega)\Big), \end{aligned} \tag{A.11}$$

The interpretation of $\boldsymbol{B^G_{j,i}}$, $j \neq i$ is similar to (16). On the other hand, $\boldsymbol{B^G_{i,i}}$ contains the service times of a complete cycle starting in Q_i. This includes the service times of all the customers that are standing behind the gate in Q_i, the service times of all the customers in $Q_{i+1}, \dots, Q_{i-1}$ that were already in the system on the arrival of the customer batch or entered the system before the next visit to Q_i, and when the server reaches Q_i again the service times of all the customers that were standing before the gate when the cycle in Q_i started.

We first focus on the batch sojourn-time of a customer batch that arrives during a visit period V_{j-1}. The batch sojourn-time of customer batch $\boldsymbol{k}$ that arrives when the server is in visit period V_{j-1} consists of the (i) residual service time in Q_{j-1}, (ii) the service of all the customers behind the gate in $Q_{j-1}, \dots, Q_i$, (iii) the service of all new customer arrivals that arrive after customer batch $\boldsymbol{k}$ in $Q_j, \dots, Q_{i-1}$ before the server reaches Q_i, (iv) switch-over times $S_{j-1}, \dots, S_{i-1}$, (v) the service of the customers in customer batch $\boldsymbol{k}$, and (vi) if $i = j-1$ also the customers before the gate in Q_i. Because incoming customers are placed before the gate when the server is in visit period V_{j-1}, we have to modify (19) to,

$$\begin{aligned} \widetilde{L}^{(V_{j-1})}(\boldsymbol{z}, \omega) = \widetilde{LB}^{(B_{j-1})}(\boldsymbol{z}) \\ \times \widetilde{B}^{PR}_{j-1}\left(\lambda - \lambda K(z_1, \dots, z_{j-2}, z_G, z_j, \dots, z_N), \omega\right). \end{aligned} \tag{A.12}$$

Then, the LST of batch sojourn-time distribution of batch $\boldsymbol{k}$ given that the server is in visit period V_{j-1} is given in the next proposition.

Proposition A.1 *The LST of the batch sojourn-time distribution of batch $\boldsymbol{k}$ conditioned that the server is in visit period V_{j-1} and the last customer in the batch will be served in Q_i is given by,*

$$\begin{aligned} \widetilde{T}^{(V_{j-1})}_{\boldsymbol{k}}(\omega) = \widetilde{L}^{(V_{j-1})}\left(\boldsymbol{B^G_{j-1,i}}, \omega + \lambda(1 - \widetilde{K}(\boldsymbol{B_{j,i-1}}))\right) \\ \times \prod_{l=j-1}^{i-1}{}' \widetilde{S}_{l,i-1}(\omega) \frac{1}{(\boldsymbol{B^G_{j-1,i}})_{j-1}} \prod_{l=j}^{i}{}' (\boldsymbol{B^*_{j,i}})_l^{k_l}. \end{aligned} \tag{A.13}$$

Proof During visit period V_{j-1} incoming customers to Q_{j-1} are placed before the gate and will be served in the next visit. Taken this into account, the same steps as in the proof of Proposition 1 can be used to derive (A.13). □

Next, we derive the LST of batch sojourn-time distribution of batch $\boldsymbol{k}$ given that the server is in switch-over period S_{j-1}. For this we modify (22) to,

$$\widetilde{L}^{(S_{j-1})}(\boldsymbol{z},\omega) = \widetilde{LB}^{(S_{j-1})}(\boldsymbol{z}) \times \widetilde{S}_{j-1}^{PR}\left(\lambda - \lambda\widetilde{K}(z_1,\ldots,z_{j-2},z_{j-1},z_j,\ldots,z_N),\omega\right). \quad \text{(A.14)}$$

Proposition A.2 *The LST of the batch sojourn-time distribution of batch $\boldsymbol{k}$ conditioned that the server is in switch-over period S_{j-1} and the last customer in the batch will be served in Q_i is given by*

$$\widetilde{T}_{\boldsymbol{k}}^{(S_{j-1})}(\omega) = \widetilde{L}^{(S_{j-1})}\left(\boldsymbol{B}^*_{\boldsymbol{j,i}},\, \omega + \lambda(1-\widetilde{K}(\boldsymbol{B_{j,i-1}}))\right) \times \prod_{l=1}^{i-j}{}' \widetilde{S}_{j+l-1,i-1}(\omega) \prod_{l=j}^{i}{}' (\boldsymbol{B}^*_{\boldsymbol{j,i}})_l^{k_l}. \quad \text{(A.15)}$$

Proof Similarly, the same steps as in the proof of Proposition 2 can be used to derive (A.15). □

From Proposition A.1 and Proposition A.2, it can be seen that the LST of the batch sojourn-time distribution of batch $\boldsymbol{k}$ conditioned on a visit/switch-over period can be decomposed into two terms;

$$\widetilde{T}_{\boldsymbol{k}}^{(V_{j-1})}(\omega) = \sum_{i=1}^{N} 1_{(\boldsymbol{k}\in\mathcal{K}_{j,i})} \widetilde{W}_i^{(V_{j-1})}(\omega) \prod_{l=j}^{i}{}' (\boldsymbol{B}^*_{\boldsymbol{j,i}})_l^{k_l}, \quad \text{(A.16)}$$

$$\widetilde{T}_{\boldsymbol{k}}^{(S_{j-1})}(\omega) = \sum_{i=1}^{N} 1_{(\boldsymbol{k}\in\mathcal{K}_{j,i})} \widetilde{W}_i^{(S_{j-1})}(\omega) \prod_{l=j}^{i}{}' (\boldsymbol{B}^*_{\boldsymbol{j,i}})_l^{k_l}, \quad \text{(A.17)}$$

where $\widetilde{W}_i^{(V_{j-1})}(\omega)$ and $\widetilde{W}_i^{(S_{j-1})}(\omega)$ can be considered as the time between the batch arrival epoch and the service completion of the last customer in Q_i that is already in the system, excluding any arrivals to Q_i after the arrival epoch and contribution of the batch.

The LST of the batch sojourn-time distribution of a specific customer batch $\boldsymbol{k}$ can now be calculated by,

$$\widetilde{T}_{\boldsymbol{k}}(\omega) = \frac{1}{E(C)} \sum_{j=1}^{N}\sum_{i=1}^{N} 1_{(\boldsymbol{k}\in\mathcal{K}_{j,i})} \left(E(V_{j-1})\widetilde{W}_i^{(V_{j-1})}(\omega) + E(S_{j-1})\widetilde{W}_i^{(S_{j-1})}(\omega)\right) \prod_{l=j}^{i}{}' (\boldsymbol{B}^*_{\boldsymbol{j,i}})_l^{k_l}. \quad \text{(A.18)}$$

Finally, we focus on the LST of the batch sojourn-time of an arbitrary batch $\widetilde{T}(.)$.

Theorem A.1 *The LST of the batch sojourn-time distribution of an arbitrary batch $\widetilde{T}(.)$, if this queue receives locally-gated service, is given by:*

$$\widetilde{T}(\omega) = \sum_{\boldsymbol{k}\in\mathcal{K}} \pi(\boldsymbol{k})\, \widetilde{T}_{\boldsymbol{k}}(\omega), \quad \text{(A.19)}$$

where $\widetilde{T}_{\boldsymbol{k}}(\omega)$ is given by (A.18). *Alternatively, we can write* (A.19) *as,*

$$\widetilde{T}(\omega) = \frac{1}{E(C)} \sum_{j=1}^{N}\sum_{i=1}^{N} \left(E(V_{j-1})\widetilde{W}_i^{(V_{j-1})}(\omega) + E(S_{j-1})\widetilde{W}_i^{(S_{j-1})}(\omega)\right) \times \pi(\mathcal{K}_{j,i})\, \widetilde{K}\left(\boldsymbol{B}^*_{\boldsymbol{j,i}}|\mathcal{K}_{j,i}\right). \quad \text{(A.20)}$$

Proof Using the definition of $\mathcal{K}_{j,i}$, the proof is almost identical to the one of Theorem 1. □

A.3 Mean value analysis

In this section, we will use MVA again to derive the mean batch sojourn-time of a specific batch and an arbitrary batch. Central in the MVA for locally-gated service is $E\left(\bar{L}_i^{(V_j,S_j)}\right)$, the mean queue-length at Q_i (excluding the potential customer currently in service) at an arbitrary epoch within visit period V_j and switch-over period S_j. First, for notation purposes we introduce θ_j as shorthand for intervisit period (V_j, S_j); the expected duration of this period $E(\theta_j)$ is given by,

$$E(\theta_j) = E(V_j) + E(S_j), \quad j = 1, \dots, N. \tag{A.21}$$

The big difference with Sect. 4.3 is that we know have to consider the customers that stand before the gate and those who stand behind. For this we introduce variables $E\left(\tilde{L}_i^{(\theta_j)}\right)$ as the expected number of customers standing before the gate the gate in Q_i during intervisit period θ_j and $E\left(\hat{L}_i^{(\theta_i)}\right)$ as the expected number of customers standing behind the gate the gate in Q_i during intervisit period θ_i. In MVA customers all incoming customers are placed before the gate, and only placed behind the gate when a visit period begins. Note this is a slight difference with Sect. A.1 where only customers arriving to the same queue that is being visited are placed before the gate. Then the mean queue-length in Q_i, $E\left(\bar{L}_i^{(\theta_j)}\right)$, given that the server is not in intervisit period θ_i, i.e. $i \neq j$, is equal to the mean number of customers standing before the gate $E\left(\tilde{L}_i^{(\theta_j)}\right)$. Otherwise, when $i = j$ the mean queue length in Q_i is the sum of the number of customers standing in front and behind the gate. Thus we can write $E\left(\bar{L}_i^{(\theta_j)}\right)$ as,

$$E\left(\bar{L}_i^{(\theta_j)}\right) = \begin{cases} E\left(\tilde{L}_i^{(\theta_j)}\right) + E\left(\hat{L}_i^{(\theta_i)}\right), & i = j, \\ E\left(\tilde{L}_i^{(\theta_j)}\right), & \text{otherwise.} \end{cases}$$

Subsequently, the mean queue-length in Q_i is given by,

$$E\left(\bar{L}_i\right) = \sum_{j=1}^{N} \frac{E(\theta_j)}{E(C)} E\left(\tilde{L}_i^{(\theta_j)}\right) + \frac{E(\theta_i)}{E(C)} E\left(\hat{L}_i^{(\theta_i)}\right), \quad i = 1, \dots, N. \tag{A.22}$$

We denote by $E(B_{j,i})$ as the the mean duration a service time B_j and its descendants before the server starts service in Q_i given that the server is currently in Q_j. Let $E(B_{j,j+1}) = E(B_j)$ be the expectation of B_j and $E(B_{j,j+2}) = E(B_j)(1 + \rho_{j+1})$ be the sum of the service time B_j and the service of all the customers that arrive in Q_{j+1} during this service. In general we can write $E(B_{j,i})$ for $i \neq j+1$ as,

$$E(B_{j,i}) = E(B_j) \prod_{l=j+1}^{i-1}{}' (1 + \rho_l), \quad i = 1, \dots, N, \, j = 1, \dots, N. \tag{A.23}$$

Finally, $E(S_{j,i})$, $E\left(B_{j,i}^R\right)$, and $E\left(S_{j,i}^R\right)$ are given by $E(B_{j,i})$ and replacing $E(B_j)$ with $E(S_j)$, $E\left(B_j^R\right)$, and $E\left(S_j^R\right)$ respectively.

Again, we consider the waiting time $E(W_i)$ of an arbitrary customer and make extensively use of Little's Law and the PASTA property. When the customer enters the system at Q_i, it has to wait for the next visit to Q_i. Even if the customer enters the system while the server is in intervisit period θ_i, the customer is placed before the gate and will only be served when the server returns to this queue in the next cycle. The average duration of the server returning to Q_i equals $E\left(\theta_{i,i-1}^R\right)$. Then at Q_i, the customer first has to wait for the service of the average number of customers $E\left(\tilde{L}_i\right) = \sum_{j=1}^{N} E(\theta_j)/E(C)\, E\left(\tilde{L}_i^{(\theta_j)}\right)$ that are in front of the customer when it arrived in the system, as well as, the service of $E(K_{ii})/2E(K_i)$ customers that arrived in the same customer batch, but are placed before the arbitrary customer in Q_i. This gives the following expression for the mean waiting time $E(W_i)$,

$$E(W_i) = E\left(\tilde{L}_i\right) E(B_i) + \frac{E(K_{ii})}{2E(K_i)} E(B_i) + E\left(\theta^R_{i,i-1}\right), \tag{A.24}$$

Applying Little's law gives,

$$E\left(\bar{L}_i\right) = \rho_i E\left(\tilde{L}_i\right) + \rho_i \frac{E(K_{ii})}{2E(K_i)} + \lambda_i E\left(\theta^R_{i,i-1}\right). \tag{A.25}$$

The next step is to derive the equations is to relate unknowns $E\left(\theta^R_{i,i-1}\right)$ to $E\left(\tilde{L}_i^{(\theta_j)}\right)$ and $E\left(\hat{L}_i^{(\theta_i)}\right)$. Consider $E\left(\theta^R_{j,i}\right)$ the expected residual duration of an intervisit period starting in θ_j and ending in θ_i given that an arbitrary customer batch just entered the system. Then with probability $E(\theta_l)/E(\theta_{j,i})$, the server is during this period in intervisit period θ_l, $l = j, \ldots, i$, and the expected residual duration until the intervisit ending of θ_i, conditioned that the server is in intervisit period θ_l, is defined as follows. First, with probability $E(V_l)/E(\theta_l)$ the customer has to wait for the server serving a customer in Q_l and switch-over period S_l and with probability $E(S_l)/E(C)$ the customer has to wait for a residual switch-over period in S_l. Also, $E\left(\hat{L}_l^{(\theta_j)}\right)$ customers are standing behind the gate in Q_l that need to be served. During this period new descendants can arrive in the system that will be served before the intervisit ending in θ_j. In addition, for each queue Q_n, $n = j+1, \ldots, i$, the expected number of customers in the Q_n given that the server is in θ_l, $E\left(\tilde{L}_n^{(\theta_l)}\right)$, and the expected number of customers that arrived in Q_n in the arbitrary customer batch $E(K_{nl})/E(K_n)$ will increase the duration of $E\left(\theta^R_{j,i}\right)$ by $E(B_{n,i+1})$. Finally, the switch-over times between Q_n to Q_{n+1} plus all its descendants that will be served before the end of the period contribute with $E(S_{n,i+1})$. Combining this gives the following expression,

$$\begin{aligned} E\left(\theta^R_{j,i}\right) = {\sum_{l=j}^{i}}' \frac{E(\theta_l)}{E(\theta_{j,i})} \Bigg(& \frac{E(V_l)}{E(\theta_l)} \left(E\left(B^R_{l,i+1}\right) + E(S_{l,i+1}) \right) \\ & + \frac{E(S_l)}{E(\theta_l)} E\left(S^R_{l,i+1}\right) + E\left(\hat{L}_l^{(\theta_l)}\right) E(B_{l,i+1}) \\ & + {\sum_{n=1}^{i-l}}' \left(\frac{E(K_{l+n,l})}{E(K_{l+n})} + E\left(\tilde{L}_{l+n}^{(\theta_l)}\right) \right) E(B_{l+n,i+1}) + E(S_{l+n,i+1}) \Bigg). \end{aligned} \tag{A.26}$$

It is now possible to set up a set of $N(N+1)$ linear equations in terms of unknowns $E\left(\tilde{L}_i^{(\theta_j)}\right)$ and $E\left(\hat{L}_i^{(\theta_i)}\right)$. First, the number of customers in Q_i before the gate given an arbitrary moment in an intervisit period starting in θ_i and ending in θ_j equals the number of Poisson arrivals during the age of this period. Since the age is in distribution equal to the residual time, the following equation holds, $i = 1, \ldots, N$, $j = 1, \ldots, N$,

$${\sum_{l=i}^{j}}' \frac{E(\theta_l)}{E(\theta_{i,j})} E\left(\tilde{L}_i^{(\theta_l)}\right) = \lambda_i E\left(\theta^R_{i,j}\right). \tag{A.27}$$

Second, by (A.24) and using Little's Law $\lambda_i E(W_i) = E\left(\bar{L}_i\right)$ into (A.25) gives, for $i = 1, 2 \ldots, N$,

$$\begin{aligned} (1-\rho_i) \sum_{j=1}^{N} \frac{E(\theta_j)}{E(C)} E\left(\tilde{L}_i^{(\theta_j)}\right) + \frac{E(\theta_i)}{E(C)} E\left(\hat{L}_i^{(\theta_i)}\right) - \rho_i \frac{E(K_{ii})}{2E(K_i)} \\ = \lambda_i E\left(\theta^R_{i,i-1}\right). \end{aligned} \tag{A.28}$$

With (A.27) and (A.28) a set of $N(N+1)$ linear equations are now defined. Solving the set of linear equations and by (A.25) and (A.24) will give the expected queue-lengths and waiting times.

It is now possible to derive the mean batch time $E(T_{\boldsymbol{k}})$ of customer batch $\boldsymbol{k}$ using (39). For this we need to calculate $E\left(T_{\boldsymbol{k}}^{(\theta_{j-1})}\right)$. When customer batch $\boldsymbol{k}$ enters the system and the server is in intervisit period θ_{j-1}, then with probability $E(V_{j-1})/E(\theta_{j-1})$ and $E(S_{j-1})/E(\theta_{j-1})$ the arriving customer batch has to wait for the residual service and a switch-over or a residual switch-over time during in which new customer can arrive that will be served before the visit completion in Q_{i-1}. Then each customer already in the system and in batch $\boldsymbol{k}$ in Q_l, $l = j-1, \ldots, i$ and their descendants will increase the batch sojourn-time. Finally, the batch also has to wait for all the switch-over times between Q_j to Q_{i-1} and all their descendants that will be served before the server reaches Q_i. This gives the following expression,

$$E\left(T_{\boldsymbol{k}}^{(\theta_{j-1})}\right) = \frac{E(V_{j-1})}{E(\theta_{j-1})}\left(E\left(B_{j-1,i}^{R}\right) + E(S_{j-1,i})\right) + \frac{E(S_{j-1})}{E(\theta_{j-1})}E\left(S_{j-1,i}^{R}\right)$$
$$+ E\left(\hat{L}_{j-1}^{(\theta_{j-1})}\right)E(B_{j-1,i})$$
$$+ \sum_{l=1}^{i-j}{}' \left(E\left(\tilde{L}_{j+l-1}^{(\theta_{j-1})}\right) + k_{j+l-1}\right)E(B_{j+l-1,i}) + E(S_{j+l-1,i})$$
$$+ \left(\left(\tilde{L}_{i}^{(\theta_{j-1})}\right) + k_i\right)E(B_i), \quad \text{(A.29)}$$

Notice that the same decomposition as (24) and (25) also holds for the expected batch sojourn-time,

$$E\left(T_{\boldsymbol{k}}^{(\theta_{j-1})}\right) = E\left(W_i^{(\theta_{j-1})}\right) + \sum_{l=1}^{i-j}{}' k_{j+l-1}E(B_{j+l-1,i}) + k_iE(B_i), \quad \text{(A.30)}$$

where $E\left(W_i^{(\theta_{j-1})}\right)$ is the expected time between the batch arrival epoch and the service completion of the last customer in Q_i that is already in the system, excluding any arrivals to Q_i after the arrival epoch.

Finally, the expected batch sojourn-time of an arbitrary customer batch is given by (41). Similarly, we can rewrite (41) by taking the expectation of $\mathcal{K}_{j,i}$ and using (A.30),

$$E(T) = \frac{1}{E(C)}\sum_{j=1}^{N}\sum_{i=1}^{N}E(\theta_j)\,\pi(\mathcal{K}_{j,i})\,(E\left(W_i^{(\theta_{j-1})}\right)$$
$$+ \sum_{l=1}^{i-j}{}' E(K_{j+l-1}|\mathcal{K}_{j,i})\,E(B_{j+l-1,i}) + E(K_i|\mathcal{K}_{j,i})\,E(B_i)).$$

B Globally-gated service

In this section the batch sojourn distribution under globally-gated service is studied in Sect. B.1, and the mean batch sojourn-times in Sect. B.2.

B.1 Batch sojourn distribution

Under the globally-gated service discipline all the customers that were present at the visit beginning of reference queue Q_1 will be served during the coming cycle. Meanwhile, customers that arrive in the system during this cycle have to wait and will be served in the next cycle. The advantage of the globally-gated service discipline is that closed-form expressions can be easily be derived for the delay distribution compared to exhaustive and locally-gated [2].

Let random variables $n_1, \ldots n_N$ denote the number of customers in the queues at the beginning of an arbitrary cycle C and let $\widetilde{C}(\omega) = E\left(e^{-\omega C}\right)$ be its LST. Then, the length of the current cycle will equal the sum of all switch-over times and the total sum of all the service times of the customers present at the beginning of the cycle. Combining this gives,

$$E\left(e^{-\omega C}|n_1, \ldots, n_N\right) = \widetilde{S}(\omega) \prod_{j=1}^{N} \widetilde{B}_j^{n_j}(\omega), \tag{B.1}$$

where $\widetilde{S}(\omega) = \prod_{j=1}^{N} \widetilde{S}_j(\omega)$. On the other hand, the length of a cycle determines the joint queue-length distribution at the beginning of the next cycle [2],

$$\begin{aligned} E\left(z_1^{n_1} \cdots z_N^{n_N}\right) &= E\left(E\left(z_1^{n_1} \cdots z_N^{n_N} | C = t\right)\right) \\ &= E\left(\exp\left(-\left(\lambda - \lambda\widetilde{K}(\boldsymbol{z})\right)t\right)\right) = \widetilde{C}\left(\lambda - \lambda\widetilde{K}(\boldsymbol{z})\right). \end{aligned} \tag{B.2}$$

With use of (B.1) and (B.2), we have

$$\begin{aligned} \widetilde{C}(\omega) &= \widetilde{S}(\omega) E\left(\widetilde{B}_1^{n_1}(\omega) \cdots \widetilde{B}_N^{n_N}(\omega)\right) \\ &= \widetilde{S}(\omega) \widetilde{C}\left(\lambda - \lambda K\left(\widetilde{B}_1(\omega), \ldots, \widetilde{B}_N(\omega)\right)\right). \end{aligned} \tag{B.3}$$

Let C_P and C_R be the past and residual time, respectively, of a cycle. We can write the LST of the joint distribution of C^P and C^R as [3],

$$\widetilde{C}^{PR}(\omega_P, \omega_R) = \frac{\widetilde{C}(\omega_R) - \widetilde{C}(\omega_P)}{E(C)(\omega_P - \omega_R)}, \tag{B.4}$$

and

$$\widetilde{C}^P(\omega_R) = \widetilde{C}^R(\omega_P) = \frac{1 - \widetilde{C}(\omega)}{\omega E(C)}. \tag{B.5}$$

Finally, let $\boldsymbol{B_{j,i}}$ be an N-dimensional vector with the LST of the service times of Q_l on elements $l = j, \ldots, i$,

$$\boldsymbol{B_{j,i}} = \left(1, \ldots, \widetilde{B}_j(\omega), \widetilde{B}_{j+1}(\omega), \ldots, \widetilde{B}_i(\omega), 1, \ldots, 1\right).$$

With the previous results, we can now derive the LST of the batch sojourn distribution of specific batch of customers.

Proposition B.1 *The LST of the batch sojourn-time distribution of batch* $\boldsymbol{k}$ *is given by,*

$$\widetilde{T}_{\boldsymbol{k}}(\omega) = \frac{1}{E(C)} \left[\frac{\widetilde{C}\left(\lambda - \lambda\widetilde{K}\left(\boldsymbol{B_{1,i}}\right)\right) - \widetilde{C}\left(\lambda - \lambda\widetilde{K}\left(\boldsymbol{B_{1,i-1}}\right) + \omega\right)}{\omega - \lambda\left(1 - \widetilde{K}\left(\boldsymbol{B_{i,i}}\right)\right)}\right] \prod_{j=1}^{i-1} \widetilde{S}_j(\omega) \times \prod_{j=1}^{i} k_i \widetilde{B}_j(\omega). \tag{B.6}$$

Proof Assume an arbitrary customer batch $\boldsymbol{k}$ where the number of customer arrivals per queue is $k_1 \geq 0, \ldots, k_i > 0$ and $k_{i+1} = 0, \ldots, k_N$. Due to the globally-gated service discipline, any arriving customer batch will be totally served in the next cycle, which implies that the customer batch will be fully served after its last customer in Q_i is served. Then, the batch sojourn-time of customer batch $\boldsymbol{k}$ is composed of; (i) the residual cycle time C^R, (ii) the service times of all customers who arrive at $Q_1, \ldots, Q_{i-1}$ during the cycle in which the new customer batch arrives, (iii) the switch-over times of the server between $Q_1, \ldots, Q_{i-1}$, (iv) the service times of all the customers who arrive at Q_i during the past part C^P of the cycle in which the customer batch arrives, and (v) the service times of all the customers in the batch at $Q_1, \ldots, Q_i$. Combining this gives,

$$T_{\boldsymbol{k}} = C^R + \sum_{j=1}^{i-1} \sum_{m=1}^{N_j(C^P+C^R)} B_{j_m} + \sum_{j=1}^{i-1} S_j + \sum_{m=1}^{N_i(C^P)} B_{i_m} + \sum_{j=1}^{i} \sum_{m=1}^{k_j} B_{j_m}, \tag{B.7}$$

where $N_j\left(C^P + C^R\right)$ denotes number of arrivals in Q_j during the past and residual time of the current cycle and $N_i\left(C^P\right)$ denotes the number of arriving customers in Q_i during C^P. Note that the cycle in which the customer batch arrives is not equal to $E\left(C\right)$, but is atypical of size $E\left(C^P\right) + E\left(C^R\right)$ [2]. By taking the LST of (B.7) we obtain,

$$\begin{aligned}\widetilde{T}_{\boldsymbol{k}}\left(\omega\right) &= \prod_{j=1}^{i-1} \widetilde{S}_j\left(\omega\right) \int_{t_P=0}^{\infty} \int_{t_R=0}^{\infty} e^{-\omega t_R} e^{-(\lambda-\lambda K(\boldsymbol{B_{1,i-1}}))(t_P+t_R)} \\ &\quad \times e^{-(\lambda-\lambda\widetilde{K}(\boldsymbol{B_{i,i}}))t_P} \mathrm{d}Pr\left(C^P < t_P,\, C^R < t_R\right) \prod_{j=1}^{i} k_j \widetilde{B}_j\left(\omega\right) \\ &= \prod_{j=1}^{i-1} \widetilde{S}_j\left(\omega\right) E\Big(\exp\Big(-\left(\lambda - \lambda\widetilde{K}\left(\boldsymbol{B_{1,i}}\right)\right) C^P \\ &\quad - \left(\lambda - \lambda\widetilde{K}\left(\boldsymbol{B_{1,i-1}}\right) + \omega\right) C^R\Big)\Big) \prod_{j=1}^{i} k_j \widetilde{B}_j\left(\omega\right),\end{aligned}$$

Using the LST of the joint distribution of C_P and C_R of (B.4), we obtain (B.6). □

We can now find the LST of the batch sojourn-time distribution of an arbitrary batch.

Theorem B.1 *The LST of the batch sojourn-time distribution of an arbitrary batch $\widetilde{T}\left(.\right)$, if this queue receives globally-gated service, is given by:*

$$\widetilde{T}\left(\omega\right) = \sum_{\boldsymbol{k}\in\mathcal{K}} \pi\left(\boldsymbol{k}\right) \widetilde{T}_{\boldsymbol{k}}\left(\omega\right), \tag{B.8}$$

where $\widetilde{T}_{\boldsymbol{k}}\left(\omega\right)$ is given by (13). *Alternatively, we can write* (B.6) *as,*

$$\widetilde{T}\left(\omega\right) = \frac{1}{E\left(C\right)} \sum_{i=1}^{N} \left[\frac{\widetilde{C}\left(\lambda - \lambda\widetilde{K}\left(\boldsymbol{B_{1,i}}\right)\right) - \widetilde{C}\left(\lambda - \lambda\widetilde{K}\left(\boldsymbol{B_{1,i-1}}\right) + \omega\right)}{\omega - \lambda\left(1 - \widetilde{K}\left(\boldsymbol{B_{i,i}}\right)\right)}\right] \prod_{j=1}^{i-1} \widetilde{S}_j\left(\omega\right) \pi\left(\mathcal{K}_{1,i}\right) \widetilde{K}\left(\boldsymbol{B_{1,i}} | \mathcal{K}_{1,i}\right). \tag{B.9}$$

Proof In case of locally-gated an incoming customer batch can only be served in the next cycle. Therefore, independently on the location of the server the last customer in the batch to be served is located in the queue that is the farthest loacted from the reference queue. Thus, we can write

$$\widetilde{T}\left(\omega\right) = \sum_{\boldsymbol{k}\in\mathcal{K}} \sum_{i=1}^{N} 1_{\left(\boldsymbol{k}\in\mathcal{K}_{1,i}\right)} \pi\left(\boldsymbol{k}\right) \widetilde{T}_{\boldsymbol{k}}\left(\omega\right).$$

Finally, by inserting (B.6) and (1) we obtain (B.9). □

B.2 Mean batch sojourn-time

In this section we determine $E\left(T_{\boldsymbol{k}}\right)$, the expected batch sojourn-time for a specific customer batch $\boldsymbol{k}$. Instead of using MVA, as was the case for exhaustive and locally-gated, we can directly calculate $E\left(T_{\boldsymbol{k}}\right)$ similar as for the mean waiting time [2]. Taking the expectation of (B.7) gives the following expression,

$$E\left(T_{\boldsymbol{k}}\right) = E\left(C^R\right) + \sum_{j=1}^{i-1} \lambda_j E\left(B_j\right)\left(E\left(C^P\right) + E\left(C^R\right)\right) + \sum_{j=1}^{i-1} E\left(S_j\right) + \rho_i E\left(C^P\right) + \sum_{j=1}^{i} k_j E\left(B_j\right). \quad \text{(B.10)}$$

What is left is to derive the mean past and residual time of the cycle time, $E\left(C_P\right)$ and $E\left(C_R\right)$. Differentiating (B.3) once and twice yields closed-form expressions for the first two moment of the cycle time,

$$E\left(C\right) = \frac{E\left(S\right)}{\left(1-\rho\right)}, \quad \text{(B.11)}$$

$$E\left(C^2\right) = \frac{1}{\left(1-\rho^2\right)}\left[E\left(S^2\right) + 2\rho E\left(S\right)E\left(C\right) + \sum_{j=1}^{N}\lambda_j E\left(B_j^2\right)E\left(C\right) + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda E\left(K_{ij}\right)E\left(B_i\right)E\left(B_j\right)E\left(C\right)\right]. \quad \text{(B.12)}$$

and the expected past and residual cycle time is given by

$$E\left(C^P\right) = E\left(C^R\right) = \frac{E\left(C^2\right)}{2E\left(C\right)} = \frac{1}{\left(1+\rho\right)}\left[\frac{E\left(S^2\right)}{2E\left(S\right)} + \frac{\rho E\left(S\right)}{\left(1-\rho\right)} + \frac{\sum_{j=1}^{N}\lambda_j E\left(B_j^2\right) + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda E\left(K_{ij}\right)E\left(B_i\right)E\left(B_j\right)}{2\left(1-\rho\right)}\right]. \quad \text{(B.13)}$$

Using (B.13), we can rewrite (B.10) as follows,

$$E\left(T_{\boldsymbol{k}}\right) = \left[1 + 2\sum_{j=1}^{i-1}\rho_j + \rho_i\right]\frac{E\left(C^2\right)}{2E\left(C\right)} + \sum_{j=1}^{i-1}E\left(S_j\right) + \sum_{j=1}^{i}k_j E\left(B_j\right). \quad \text{(B.14)}$$

Finally, we can derive $E\left(T\right)$ the expected batch sojourn-time of an arbitrary customer batch. Multiplying $E\left(T_{\boldsymbol{k}}\right)$ with all possible realizations of $\boldsymbol{k}$ and using $\mathcal{K}_{1,i}$ gives,

$$\begin{aligned}
E\left(T\right) &= \sum_{i=1}^{N}\sum_{\boldsymbol{k}\in\mathcal{K}_{1,i}}\pi\left(\boldsymbol{k}\right)E\left(T_{\boldsymbol{k}}\right) \\
&= \sum_{i=1}^{N}\pi\left(\mathcal{K}_{1,i}\right)\left(\left[1 + 2\sum_{l=1}^{i-1}\rho_l + \rho_i\right]\frac{E\left(C^2\right)}{2E\left(C\right)} + \sum_{j=1}^{i-1}E\left(S_j\right)\right) \\
&\quad + \sum_{j=1}^{N}E\left(K_j\right)E\left(B_j\right) \\
&= \frac{E\left(C^2\right)}{2E\left(C\right)} + \sum_{i=1}^{N}\left(\rho_i\frac{E\left(C^2\right)}{E\left(C\right)} + E\left(S_i\right)\right)\cdot\left(1 - \sum_{j=1}^{i}\pi\left(\mathcal{K}_{1,j}\right)\right) \\
&\quad + \rho_i\frac{E\left(C^2\right)}{2E\left(C\right)}\pi\left(\mathcal{K}_{1,i}\right) + E\left(K_i\right)E\left(B_i\right).
\end{aligned}$$

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