Online Appendix: Proofs

Candidates' preferences and the government budget constraint

The general form of the budget constraint is $\Pi(\mathbf{q}) + t\varpi - r \geq 0$. The profit component is,

$$\Pi(\mathbf{q}) = p[y^P(\mathbf{q}) + y^R(p)] - K \tag{1}$$

where y^k is the demand for good y with $k = P, R$, and K represents the fixed costs.

The first step to reach the candidates' objective function and the government budget constraint is defining the demand for the regulated good by the poor. Recall the poor spend all the income on y. The total (before tax and transfer) income of the poor group P is,

$$W^0(\mathbf{q}) = \int_{\omega^-}^{\omega^0(\mathbf{q})} \omega dF(\omega) = \int_{\omega^-}^{\omega^0(\mathbf{q})} \omega \frac{dF(\omega)}{d\omega} d\omega \tag{2}$$

By the FTC, the partial derivative of W^0 w.r.t. the policy component z ($z = r, t, p$) is,

$$W_z^0 = \frac{\partial}{\partial z}\Big[\int_{\omega^-}^{\omega^0(z)} \omega \frac{dF(\omega)}{d\omega} d\omega\Big] = \frac{\partial \omega^0(z)}{\partial z}\Big[\omega^0(z) f(\omega^0(z))\Big] \tag{3}$$

Replacing the expression for the income threshold $\omega^0(\mathbf{q}) = \frac{\psi'^{-1}(p)p - r}{1-t}$ and opening by policy component,

$$\begin{aligned}
W_r^0 &= (1-t)^{-2}[\psi'^{-1}(p)p - r] f(\omega^0(\mathbf{q}))(-1) && < 0 \\
W_t^0 &= (1-t)^{-3}[\psi'^{-1}(p)p - r]^2 f(\omega^0(\mathbf{q})) && > 0 \\
W_p^0 &= (1-t)^{-2}[(\psi'^{-1})'(p)p + \psi'^{-1}(p)][\psi'^{-1}(p)p - r] f(\omega^0(\mathbf{q})) && > 0
\end{aligned} \tag{4}$$

It is always the case that $W_t^0 > 0$. The total income of group P is decreasing in r ($W_r^0 < 0$) whenever $\psi'^{-1}(p)p > r$. This condition is satisfied for all the strictly positive values of the poverty line. Besides, $W_p^0 > 0$ also requires the marginal revenue (w.r.t. p) of the regulated good provision to group R to be strictly positive ($(\psi'^{-1})'(p)p + \psi'^{-1}(p) > 0$). This happens for every tariff below the monopoly one, since for tractability we assume zero marginal cost.

The indirect utility function of citizens in group R is $v(p) + (1-t)\omega + r$. Then, by the Roy's identity, the demand for y, y^R, is $-v'(p)$. As a result, the profit function is,

$$\Pi(\mathbf{q}) = (1-t)\int_{\omega^-}^{\omega^0(\mathbf{q})} \omega dF(\omega) + a^P(\mathbf{q})r - a^R(\mathbf{q})pv'(p) - K \tag{5}$$

with $a^P(\mathbf{q}) = F(\omega^0(\mathbf{q}))$ and $a^R(\mathbf{q}) = 1 - a^P(\mathbf{q})$. In this way, the budget constraint is given by,

$$B(\mathbf{q}) = (1-t)\int_{\omega^-}^{\omega^0(\mathbf{q})} \omega dF(\omega) + a^P(\mathbf{q})r - a^R(\mathbf{q})pv'(p) - K + t\overline{\omega} - r \geq 0 \quad (6)$$

Then, the preferences of a generic candidate c are represented by the following indirect utility function,

$$\begin{aligned} V(\mathbf{q};\omega^c,\gamma^c) &= v(p) + (1-t)\omega^c + r \\ &+ \gamma^c\Big[(1-t)\int_{\omega^-}^{\omega^0(\mathbf{q})} \omega dF(\omega) + a^P(\mathbf{q})r - a^R(\mathbf{q})pv'(p) - K\Big] \end{aligned} \quad (7)$$

The budget constraint changes with the policy components as follows,

$$\begin{aligned} B_r &= -(1-F(\omega^0)) - p\frac{f(\omega^0)}{1-t}(v'(p) + \psi'^{-1}(p)) \\ B_p &= -(1-F(\omega^0))(pv''(p) + v'(p)) \\ &+ p\frac{f(\omega^0)}{1-t}[(\psi'^{-1})'(p)p + \psi'^{-1}(p)](v'(p) + \psi'^{-1}(p)) \\ B_t &= \overline{\omega} - \int_{\omega^-}^{\omega^0(q)} \omega dF(\omega) \\ &+ p\frac{f(\omega^0)}{(1-t)^2}[\psi'^{-1}(p)p - r](v'(p) + \psi'^{-1}(p)) \end{aligned} \quad (8)$$

In turn, the change in generic candidate c's objective function when the policy components change is given by,

$$\begin{aligned} V_r^c &= 1 + \gamma^c F(\omega^0) - \gamma^c p\frac{f(\omega^0)}{1-t}(v'(p) + \psi'^{-1}(p)) \\ V_p^c &= v'(p) - \gamma^c(1-F(\omega^0))(pv''(p) + v'(p)) \\ &+ \gamma^c p\frac{f(\omega^0)}{1-t}[(\psi'^{-1})'(p)p + \psi'^{-1}(p)](v'(p) + \psi'^{-1}(p)) \\ V_t^c &= -\omega^c - \gamma^c \int_{\omega^-}^{\omega^0(q)} \omega dF(\omega) \\ &+ \gamma^c p\frac{f(\omega^0)}{(1-t)^2}[\psi'^{-1}(p)p - r](v'(p) + \psi'^{-1}(p)) \end{aligned} \quad (9)$$

V_r in 9 has four components: the increase in candidates' utility from receiving a larger r; the partial increase in the profits because those citizens still in the group P demand more of y; the negative effect on the profits for group

P shrinks; and the positive effect on the profits because of the group R's expansion. Analogous interpretations apply to V_p and V_t.

Next, we take advantage of the fact that the demand for good y is the same for every citizen in R to simplify expressions 8 and 9. Consider the consumer problem of a citizen in group R,

$$(x^*, y^*) = \underset{(x,y)\in R^2_+}{argmax} \quad \{\psi(y) + x \mid py + x \leq (1-t)\omega + r\}$$

The solution is,

$$\begin{aligned} y^* &= \psi'^{-1}(p) \\ x^* &= (1-t)\omega + r - p\psi'^{-1}(p) \end{aligned}$$

The corresponding indirect utility function,

$$\psi(\psi'^{-1}(p)) - p\psi'^{-1}(p) + (1-t)\omega + r$$

Redefining the indirect utility function as $v(p) + (1-t)\omega + r$,

$$v'(p) + \psi'^{-1}(p) = (\psi'^{-1}(p))'\Big(\psi'(\psi'^{-1}(p)) - p\Big) = 0$$

since $\psi'(\psi'^{-1}(p)) = \psi'(y^*) = p$ (i.e., by envelope). Applying this result to the equations in 8 and 9, we obtain a set of first partial derivatives of B and V^c w.r.t. the policy components as follows,

$$\begin{aligned} B_r &= -(1 - F(\omega^0)) \\ B_p &= -(1 - F(\omega^0))(pv''(p) + v'(p)) \\ B_t &= \overline{\omega} - \int_{\omega^-}^{\omega^0(q)} \omega dF(\omega) \\ V_r^c &= 1 + \gamma^c F(\omega^0) \\ V_p^c &= v'(p) - \gamma^c(1 - F(\omega^0))(pv''(p) + v'(p)) \\ V_t^c &= -\omega^c - \gamma^c \int_{\omega^-}^{\omega^0(q)} \omega dF(\omega). \end{aligned} \tag{10}$$

Benchmark: A community with no poor

Lemma A.1 (only in appendix) (Zero poverty and the median income)**.** *When there are no poor in the society, a Condorcet winner exists and coincides with the policy that the median income voter prefers.*

Proof. When there are no poor in the society, the demand for the regulated good is $-v'(p)$, equal for all the voters in $\mathbb{V}$. The binding budget constraint is,

$$t\overline{\omega} - pv'(p) - K = r$$

Then, the indirect utility function of a generic voter ω is given by,

$$V(p,r;\omega,0) = v(p) + r + \omega\Big[1 - \overline{\omega}^{-1}(pv'(p) + r + K)\Big]$$

Let ω^m be the median income voter and $\mathbf{q}_m = (p_m, r_m)$ her preferred policy, that satisfies $-p_m v'(p_m) \leq \omega^-(1 - t_m) + r_m$, where ω^- is the lowest level of income in the society. This last condition restricts the admission set of policies to $\omega^0(\mathbf{q}_m) \leq \omega^-$. Similarly, consider any alternative policy $\mathbf{q}_z$ such that $\omega^0(\mathbf{q}_z) \leq \omega^-$. Thus citizen ω will prefer $\mathbf{q}_z$ to $\mathbf{q}_m$ if and only if,

$$V(p_m, r_m; \omega, 0) < V(p_z, r_z; \omega, 0)$$

$$v(p_m) + r_m + \omega - \frac{\omega}{\overline{\omega}}(p_m v'(p_m) + r_m + K) < v(p_z) + r_z + \omega - \frac{\omega}{\overline{\omega}}(p_z v'(p_z) + r_z + K)$$

$$v(p_m) + r_m - (v(p_z) + r_z) < \frac{\omega}{\overline{\omega}}\Big[p_m v'(p_m) + r_m - (p_z v'(p_z) + r_z)\Big]$$

Since $\mathbf{q}_m$ is citizen ω^m's preferred policy, for every $q_z \neq q_m$ it must be,

$$\frac{\omega^m}{\overline{\omega}}\Big[p_m v'(p_m) + r_m - (p_z v'(p_z) + r_z)\Big] \leq v(p_m) + r_m - (v(p_z) + r_z)$$

This result opens three possibilities:

(*i*) $p_m v'(p_m) + r_m - (p_z v'(p_z) + r_z) > 0$. Then, $V(p_m, r_m; \omega, 0) \geq V(p_z, r_z; \omega, 0)$ for all $\omega \leq \omega^m$, so that at least half of the community prefers $\mathbf{q}_m$ to $\mathbf{q}_z$.

(*ii*) $p_m v'(p_m) + r_m - (p_z v'(p_z) + r_z) < 0$. Then, $V(p_m, r_m; \omega, 0) \geq V(p_z, r_z; \omega, 0)$ for all $\omega \geq \omega^m$ so that at least half of the community prefers $\mathbf{q}_m$ to $\mathbf{q}_z$.

(*iii*) $p_m v'(p_m) + r_m - (p_z v'(p_z) + r_z) = 0$. Then, by the budget constraint, $t_m = t_z$. Thus $v(p_m) + r_m \geq v(p_z) + r_z$, so that all $\omega \in \Omega$ weakly prefer $\mathbf{q}_m$ to $\mathbf{q}_z$.

Hence, $\mathbf{q}_m$ is the Condorcet winner. □

Proposition A.1 (only in appendix) (One-candidate equilibrium and the median income voter)**.** *Any one-candidate equilibrium satisfies,*

(*a*) *suppose that all the potential candidates $i \in \mathbb{C}^0$ for whom $\delta \leq V(\mathbf{q_i};, \omega^i, \gamma^i) - V(\mathbf{q_0}; \omega^i, \gamma^i)$ and $\omega^0(\mathbf{q_i}) \leq \omega^-$ have the same income. Further assume $\omega^0(\mathbf{q_0}) \leq \omega^-$. Then, the only candidate is the one who values the profits the least regardless of the median voter's location.*

(*b*) *suppose a and b in $\mathbb{C}^0$ are the two candidates who rank highest in the order of preferences of the median income voter ω^m; with $\gamma^a = \gamma^b$, $\omega^a < \overline{\omega} < \omega^b$, $\omega^0(\mathbf{q}_a) \leq \omega^-$ and $\omega^0(\mathbf{q}_b) \leq \omega^-$. Then, $r_a > r_b$. Furthermore, b (a) is the only candidate if and only if $\omega^m >$ ($<$) $\overline{\omega}$, where $\overline{\omega}$ is the mean income.*

Proof. **Part (a)**. By contradiction. Let the profile of entry actions $\alpha = (\alpha_i)_{i\in\mathbb{C}^0}$ ($\alpha_i \in \{0, 1\}$) be a Nash equilibrium of the entry game that results in the set of self-selected candidates $\mathbb{E}(\alpha) = \{a\}$ with a of type (ω^a, γ^a). Suppose there exists another potential candidate $b \in \mathbb{C}^0$ of type $\omega^b = \omega^a$ and $\gamma^b < \gamma^a$. We claim that α cannot be an equilibrium.

We first prove the tariff is increasing in the candidates' value of the profits, and the tax rate is weakly decreasing in candidates' income. To do this, we clear r from the binding constraint. Then, we replace the expression for r in the indirect utility function of a generic $i \in \mathbb{C}$,

$$S(p, \gamma^i) + T(t, \omega^i) = v(p) - (1 + \gamma^i)(pv'(p) + K) + (1 - t)\omega^i + t\overline{\omega} \tag{11}$$

In order to prove monotonicity, we take the mixed second order partial derivative of $S(\cdot)$ w.r.t. p and γ:

$$\frac{\partial S^2}{\partial p \partial \gamma} = -(pv''(p) + v'(p)) > 0 \tag{12}$$

Condition 12 is satisfied for every price strictly below the monopoly one, $p < p_M(0)$ (the profits are increasing in price). Next, we take the mixed second order partial derivative of $T(\cdot)$ w.r.t. t and ω:

$$\frac{\partial T^2}{\partial t \partial \omega} = -1 < 0 \tag{13}$$

Then, t_i is weakly decreasing in ω^i. We just proved that $p_b < p_a$ and $t_b = t_a$.

We further claim that b has an incentive to enter. The expected payoff of b from deviating by entering instead of staying out is,

$$P^b(\{a, b\})u_{bb} + \Big(1 - P^b(\{a, b\})\Big)u_{ab} - \delta \tag{14}$$

where $u_{bb} = V(\mathbf{q_b}; \omega^b, \gamma^b)$, $u_{ab} = V(\mathbf{q_a}; \omega^b, \gamma^b)$, and $P^b(\{a, b\}$ is the probability of winning of b when both b and a run in elections. $P^b(\{a, b\}) = 1$ if and only if,

$$\begin{aligned} v(p_b) + (1 - t_b)\omega^m + r_b &> v(p_a) + (1 - t_a)\omega^m + r_a \\ v(p_b) + r_b &> v(p_a) + r_a \\ v(p_b) - p_b v'(p_b) &> v(p_a) - p_a v'(p_a) \\ \psi({\psi'}^{-1}(p_b)) &> \psi({\psi'}^{-1}(p_a)) \end{aligned} \tag{15}$$

The third inequality results from the budget constraint. The fourth from $v(p) = \psi({\psi'}^{-1}(p)) - p{\psi'}^{-1}(p)$ and $-v'(p) = {\psi'}^{-1}(p)$. The inequality is satisfied for $p_b < p_a$. Hence, since $P^b(\{a, b\}) = 1$, for δ sufficiently small, b will enter

the competition. Therefore, a running unopposed cannot be an equilibrium.
Part (b).
If $\gamma^a = \gamma^b$ and $\omega^a < \overline{\omega} < \omega^b$; then, from Part a., $p_a = p_b$ and $t_a > t_b$. As a consequence, the difference in the optimal income transfer between candidates a and b is given by $r_a - r_b = \overline{\omega}(t_a - t_b) > 0$; and therefore, $r_a > r_b$. By Lemma A.1, b is the only candidate if and only if she is the preferred option for the median income voter, i.e., $(1 - t_b)\omega^m + t_b\overline{\omega} > (1 - t_a)\omega^m + t_a\overline{\omega}$. That is, $\omega^m > \overline{\omega}$. Therefore, the median income voter will prefer b (a) to a (b) if and only if, $\omega^m > (<)\,\overline{\omega}$. □

Lemma A.2 (only in appendix) (Candidates' vote share: The indifference line)**.** *For any two feasible candidates i and k, there exists a unique pair comprised of a vector $\mathbf{s}(ik)$ in $\Omega \times \Gamma$ and a scalar $c(ik)$ that satisfies,*

$$V(\mathbf{q}_i;\omega,\gamma) \gtreqless V(\mathbf{q}_k;\omega,\gamma) \text{ if and only if } [s(ik)_1, s(ik)_2]\cdot(\omega,\gamma) \gtreqless c(ik)$$

Furthermore, $\mathbf{s}(ik)$ and $c(ik)$ are fully defined by the line segment in $\Omega \times \Gamma$ given by,

$$V(\mathbf{q}_i;\omega,\gamma) - V(\mathbf{q}_k;\omega,\gamma) = 0;$$

namely the indifference line $I\{i,k\}$. The intersection of $I\{i,k\}$ with the space of voters determines the vote shares of candidates i and k.

Proof. Consider the generic form of the indirect utility function,

$$V(\mathbf{q};\omega,\gamma) = v(p) + (1-t)\omega + r + \gamma\Pi(p) \tag{16}$$

Note that, since there are no poor, the profit function $\Pi(p)$ depends only on the tariff (not on transfer r). Clearing the transfer from the binding budget constraint, we get $r = \Pi(p) + t\overline{\omega}$, with $\Pi(p) = -(pv'(p) + K)$. Replacing in 16,

$$V(\mathbf{q};\omega,\gamma) = v(p) + (1-t)\omega - (pv'(p)+K) + t\overline{\omega} + \gamma[-(pv'(p)+K)]$$

Now, we fix two candidates i and k in the space of feasible candidates $\mathbb{C}^A$, and obtain the expression for $V(\mathbf{q}_i;\cdot) - V(\mathbf{q}_k;\cdot) = 0$ as follows,

$$\begin{aligned}V(\mathbf{q}_i;\omega,\gamma) - V(\mathbf{q}_k;\omega,\gamma) =& v(p_i) - v(p_k) + (-p_iv'(p_i) + p_kv'(p_k)) + \overline{\omega}(t_i - t_k)\\ & -\omega(t_i - t_k) + \gamma(-p_iv'(p_i) + p_kv'(p_k)) = 0\end{aligned}$$

This equation is linear in both ω and γ. Therefore, it represents a line $I\{i,k\} \subset \Omega \times \Gamma \equiv \mathbb{V} \cup \mathbb{C}^A$. $I\{i,k\}$ separates the space $\mathbb{V} \cup \mathbb{C}^A$ into two convex disjoint half-spaces. $I\{i,k\}$ fully defines the pair $\Big(\mathbf{s}(ik), c(ik)\Big)$, $\mathbf{s}(ik) \equiv (s(ik)_1, s(ik)_2)$, as follows,

$$\begin{aligned} c(ik) &= v(p_i) - v(p_k) + (-p_i v'(p_i) + p_k v'(p_k)) + \overline{\omega}(t_i - t_k) \\ s(ik)_1 &= t_i - t_k \\ s(ik)_2 &= p_i v'(p_i) - p_k v'(p_k) \end{aligned} \tag{17}$$

By the proof of Proposition A.1, the indirect utilities of candidates are separable in price and tax rate. In fact, the FOCs of candidates' maximisation problems are $\gamma[p^* v''(p^*) + v'(p^*)] + p^* v''(p^*) = 0$ and t^* equals the infimum or the supremum depending on $(\omega - \overline{\omega})$ being positive or negative. As the optimal policies of candidates i and k, and the average income $\overline{\omega}$ fully parametrise the vector and the scalar, for any pair (i,k), $I\{i,k\}$ is unique. □

Corollary A.1 (only in appendix) (Slope of the indifference line)**.** *Fix a pair of candidates (i,k). Suppose $I(\{i,k\})$ intersects the space of voters at $\omega^* \in int(\mathbb{V})$, and $\omega_i < \overline{\omega} < \omega_k$. Then, the slope of $I(\{i,k\})$ is negative (positive) iff $\omega^* > (<) \overline{\omega}$.*

Proof. The assumption that $I(\{i,k\})$ intersects the voters' line at $\omega^* \in int(\mathbb{V})$ is equivalent to $(\omega^*, 0) \in I(\{i,k\})$; then, $I(\{i,k\})$ is defined by,

$$\begin{aligned} v(p_i) - p_i v'(p_i) - (v(p_k) - p_k v'(p_k)) + \overline{\omega}(t_i - t_k) &= \omega^*(t_i - t_k) \\ v(p_i) - p_i v'(p_i) - (v(p_k) - p_k v'(p_k)) + \overline{\omega}(t_i - t_k) &= \\ \omega(t_i - t_k) - \gamma(-p_i v'(p_i) &+ p_k v'(p_k)) \end{aligned} \tag{18}$$

Now we name the components of 18 as follows,

$$\begin{aligned} A(i,k) &\equiv v(p_i) - p_i v'(p_i) - (v(p_k) - p_k v'(p_k)) + \overline{\omega}(t_i - t_k) \\ B(i,k) &\equiv t_i - t_k \\ D(i,k) &\equiv -p_i v'(p_i) + p_k v'(p_k) \end{aligned} \tag{19}$$

$I(\{i,k\})$ can be expressed as $A(i,k) = \omega B(i,k) - \gamma D(i,k)$. Rearranging,

$$\gamma = -\frac{A(i,k)}{D(i,k)} + \frac{B(i,k)}{D(i,k)}\omega \tag{20}$$

By $\omega_i < \overline{\omega} < \omega_k$ and the preferences separability, $t_i > t_k$. That is, $B(i,k) > 0$. Since the indifference line intersects the voters space at $\omega^* \in int(\mathbb{V})$,

$$\begin{aligned} v(p_i) + (1 - t_i)\omega^* + r_i &= v(p_k) + (1 - t_k)\omega^* + r_k \\ v(p_i) - v(p_k) + r_i - r_k &= (t_i - t_k)\omega^* \\ v(p_i) - v(p_k) - p_i v'(p_i) + p_k v'(p_k) &= (t_i - t_k)(\omega^* - \overline{\omega}) \\ \psi(\psi'^{-1}(p_i)) - \psi(\psi'^{-1}(p_k)) &= (t_i - t_k)(\omega^* - \overline{\omega}) \end{aligned} \tag{21}$$

If $\omega^* > \overline{\omega}$, then $p_i < p_k$, $D(i,k) < 0$, and the slope of $I(\{i,k\})$ is negative. If $\omega^* < \overline{\omega}$, then $p_i > p_k$, $D(i,k) > 0$, and the slope of $I(\{i,k\})$ is positive. $\square$

Lemma 1 (Two-candidate equilibria: Incentive compatibility)**.** *Any equilibrium with two candidates (i and k such that $\omega^0(\mathbf{q}_i) \leq \omega^-$ and $\omega^0(\mathbf{q}_k) \leq \omega^-$) satisfies,*

(*a*) *the indifference line intersects the space of voters at the median income, $(\omega^m, 0) \in I(\{i,k\})$;*

(*b*) *if $\omega_i < \overline{\omega} < \omega_k$ and $\omega^m > \overline{\omega}$, then $\gamma^i < \gamma^k$;*

(*c*) *there exist two lines, $I^-(\{i,k\})$ and $I^+(\{i,k\})$ parallel to $I(\{i,k\})$, that define the minimum horizontal distance between the candidates' locations. This distance is given by $\frac{4\delta}{|t_i - t_k|}$.*

Proof. **Part (a).** By contradiction. Suppose the median income voter strictly prefers one candidate. By Lemma A.2, for any two candidates i and k in $\mathbb{C}^A$, there is a unique line $I(\{i,k\})$ that defines the vote shares of i and k. Therefore, $(\omega^m, 0) \notin I(\{i,k\})$. The proof of Lemma A.1 shows that voters' preferences satisfy the Gorman polar form; hence, the fraction of voters who prefer the same candidate as the median income voter ω^m must be greater than the fraction that prefers the other candidate; i.e., either $\Phi^i(\{i,k\}) > \frac{1}{2}$ or $\Phi^k(\{i,k\}) > \frac{1}{2}$ (in terms of probability of winning, either $P^k(\{i,k\}) = 0$ or $P^i(\{i,k\}) = 0$). In this way, for any $\delta > 0$, the strategy of entering for the candidate with zero probability of winning is strictly dominated; and therefore, it cannot be a Nash equilibrium of the entry game. As a result, any equilibrium with candidates i and k must satisfy $(\omega^m, 0) \in I(\{i,k\})$; or its equivalent, $P^i(\{i,k\}) = P^k(\{i,k\}) = \frac{1}{2}$.
Part (b). It follows from the proof of Corollary A.1.
Part (c). Any two-candidate equilibrium must satisfy the following system of Incentive Compatibility constraints (IC):

$$\begin{cases} 1/2(u_{ii} - u_{ki}) \geq \delta \\ 1/2(u_{kk} - u_{ik}) \geq \delta \end{cases} \tag{22}$$

where u_{ab}, $a = i,k$ and $b = i,k$, is the indirect utility of candidate a that results from the implementation of the policy preferred by candidate b. These constraints define two disjoint sub-spaces in $\mathbb{V} \cup \mathbb{C}^A$. Using the labelling of Corollary A.1,

$$\begin{aligned} A(i,k) - \omega^i B(i,k) + \gamma^i D(i,k) &\geq \quad 2\delta \\ A(i,k) - \omega^k B(i,k) + \gamma^k D(i,k) &\leq -2\delta \end{aligned} \tag{23}$$

where $A(i,k)$, $B(i,k)$, and $D(i,k)$ are defined in equations 19.

We construct the lines $I^+(\{i,k\})$ and $I^-(\{i,k\})$ when the constraints in 23 are binding as follows (in slope-intercept form),

$$\begin{aligned} \gamma^i &= \frac{2\delta - A(i,k)}{D(i,k)} + \frac{B(i,k)}{D(i,k)}\omega^i \\ \gamma^k &= \frac{-2\delta - A(i,k)}{D(i,k)} + \frac{B(i,k)}{D(i,k)}\omega^k \end{aligned} \tag{24}$$

The horizontal distance between the two IC's; i.e. the distance when $\gamma^i = \gamma^k$, is $\frac{4\delta}{|B(i,k)|}$ with $B(i,k) \equiv t_i - t_k$. Furthermore, from Part a., in any equilibrium with candidates i and k, $I^m(\{i,k\})$ must intersect the set of voters at the median income $(\omega^m, 0)$. This is equivalent to,

$$\begin{aligned} &A(i,k) - \omega B(i,k) + \gamma D(i,k) = 0 \\ &A(i,k) - \omega^m B(i,k) = 0 \end{aligned} \tag{25}$$

Since $A(i,k)$, $B(i,k)$, and $D(i,k)$ are the same for $I^m(\{a,b\})$, $I^+(\{a,b\})$, and $I^-(\{a,b\})$, the three lines are parallel. Also, from 23, one candidate must be located to the right of $I^+(\{i,k\})$ and the other to the left of $I^-\{i,k\}$. □

Proposition 1 (Inequality and the composition of redistribution: the zero-poverty case)**.** *Any two-candidate equilibrium in a society with no poor citizens satisfies,*

- (a) *if $\omega^m < (>)\,\overline{\omega}$, then the slope of the indifference line is increasing (decreasing) in the distance $|\omega^m - \overline{\omega}|$;*
- (b) *for any continuous change in the income distribution such that the new distribution F' satisfies $F(x) < F'(x)$ with $x = F^{-1}(\frac{1}{2})$, the new candidate, say a, will prefer both lower (higher) income transfer and lower (higher) price of the regulated good if $-p_a v'(p_a) - r_a > (<) - p_{-a}v'(p_{-a}) - r_{-a}$, where $-a$ indicates the candidate other than a.*

Proof. **Part (a)**. By Lemma 1, in a two-candidate equilibrium with candidates i and k, the slope of $I^m(\{i,k\})$ is given by,

$$Slope(I^m(\{i,k\})) = \frac{B(i,k)}{D(i,k)} = \frac{t_i - t_k}{-p_i v'(p_i) + p_k v'(p_k)} \tag{26}$$

By the budget constraints,

$$\begin{aligned} -p_i v'(p_i) &= r_i - t_i\overline{\omega} + K \\ p_k v'(p_k) &= -r_k + t_k\overline{\omega} - K \end{aligned} \tag{27}$$

If we substitute the expressions in the slope of the indifference line,

$$Slope(I^m(\{i,k\})) = \frac{t_i - t_k}{r_i - t_i\overline{\omega} - r_k + t_k\overline{\omega}} \tag{28}$$

Assume t_i and t_k do not change. Then, taking the partial derivative of the Slope w.r.t. $\overline{\omega}$,

$$\begin{aligned} \frac{\partial}{\partial\overline{\omega}} Slope(I^m(\{i,k\})) &= -\frac{(t_i - t_k)(-t_i + t_k)}{(r_i - t_i\overline{\omega} - r_k + t_k)^2} \\ &= \frac{(t_i - t_k)^2}{(r_i - t_i\overline{\omega} - r_k + t_k)^2} > 0 \quad \text{for all } t_i \neq t_k \end{aligned} \tag{29}$$

For both, a negative slope ($\omega^m > \overline{\omega}$) and a positive slope ($\omega^m < \overline{\omega}$), $I^m(\{i,k\})$ moves counterclockwise. If ($\omega^m > \overline{\omega}$), then an increase in $\overline{\omega}$ means shorter distance $|\omega^m - \overline{\omega}|$. If ($\omega^m < \overline{\omega}$), then an increase in $\overline{\omega}$ means longer distance $|\omega^m - \overline{\omega}|$.

Part (b). By Lemma 1, any two-candidate equilibrium (with candidates i and k) satisfies $I(\{i,k\}) \cap \mathbb{V} = (\omega^m, 0)$; i.e., $A(i,k;\overline{\omega}) - \omega^m B(i,k) = 0$. This feature and the budget constraints of the candidates result in the following system of equations,

$$\begin{aligned} &v(p_i) - p_i v'(p_i) - (v(p_k) - p_k v'(p_k)) + \overline{\omega}(t_i - t_k) - \omega^m(t_i - t_k) = 0 \\ &t_i - \overline{\omega}^{-1}\Big(r_i + K + p_i v'(p_i)\Big) = 0 \\ &t_k - \overline{\omega}^{-1}\Big(r_k + K + p_k v'(p_k)\Big) = 0 \end{aligned} \tag{30}$$

Taking differentials,

$$\begin{aligned} &- p_i v''(p_i)dp_i + p_k v''(p_k)dp_k + (\overline{\omega} - \omega^m)(dt_i - dt_k) + (t_i - t_k)(d\overline{\omega} - d\omega^m) = 0 \\ &dt_i - \overline{\omega}^{-2}\Big[\overline{\omega}dr_i - r_i\overline{\omega} + \overline{\omega}dK - Kd\overline{\omega} + \overline{\omega}(p_i v''(p_i) + v'(p_i))dp_i - p_i v'(p_i)d\overline{\omega}\Big] = 0 \\ &dt_k - \overline{\omega}^{-2}\Big[\overline{\omega}dr_k - r_k\overline{\omega} + \overline{\omega}dK - Kd\overline{\omega} + \overline{\omega}(p_k v''(p_k) + v'(p_k))dp_k - p_k v'(p_k)d\overline{\omega}\Big] = 0 \end{aligned}$$

We want to assess the change in the policy of one candidate (say a, $a = i,k$) for a fixed policy of the other candidate $dr_{-a} = dp_{-a} = dK = d\overline{\omega} = 0$. The subscript $-a$ denotes the candidate other than a. Then, by the system of differentials above, $dt_{-a} = 0$. These conditions result in the following system of equations,

$$\begin{aligned} &p_a v''(p_a)dp_a - (\overline{\omega} - \omega^m)dt_a - (t_{-a} - t_a)d\omega^m = 0 \\ &dt_a = \overline{\omega}^{-2}\Big[\overline{\omega}dr_a + \overline{\omega}(p_a v''(p_a) + v'(p_a))dp_k\Big] \\ &t_{-a} - t_a = \overline{\omega}^{-1}\Big[- p_a v'(p_a) - r_a + p_{-a}v'(p_{-a}) + r_{-a}\Big] \end{aligned} \tag{31}$$

Solving this system for a fixed size of the government, i.e. $dt_a = 0$, we get,

$$\begin{aligned}\frac{dr_a}{d\omega^m} &= [-p_a v'(p_a) - r_a + p_{-a} v'(p_{-a}) + r_{-a}] \times \left[\frac{-p_a v''(p_a) - v'(p_a)}{p_a v''(p_a)}\right] \\ \frac{dp_a}{d\omega^m} &= [-p_a v'(p_a) - r_a + p_{-a} v'(p_{-a}) + r_{-a}] \times \left[\frac{1}{p_a v''(p_a)}\right]\end{aligned} \tag{32}$$

with both $\left[\frac{-p_a v''(p_a) - v'(p_a)}{p_a v''(p_a)}\right]$ and $\left[\frac{1}{p_a v''(p_a)}\right]$ positive. Then,

$$\begin{array}{llll} \frac{dr_a}{d\omega^m} > 0 & \text{and} \quad \frac{dp_a}{d\omega^m} > 0 & \text{iff} & -p_a v'(p_a) - r_a > -p_{-a} v'(p_{-a}) - r_{-a} \text{ and,} \\ \frac{dr_a}{d\omega^m} < 0 & \text{and} \quad \frac{dp_a}{d\omega^m} < 0 & \text{iff} & -p_a v'(p_a) - r_a < -p_{-a} v'(p_{-a}) - r_{-a}. \end{array}$$

Now, as the proposition states, consider a continuous change to $F'(x)$ such that $F(x) < F'(x)$ with $x = F^{-1}(\frac{1}{2})$, then $F'[F^{-1}(\frac{1}{2})] > 1/2$. As a consequence, in order to re-establish the equilibrium it must be that $d\omega^m < 0$. Therefore,

$$\begin{array}{llll} dr_a < 0 & \text{and} \quad dp_a < 0 & \text{iff} & -p_a v'(p_a) - r_a > -p_{-a} v'(p_{-a}) - r_{-a} \text{ and,} \\ dr_a > 0 & \text{and} \quad dp_a > 0 & \text{iff} & -p_a v'(p_a) - r_a < -p_{-a} v'(p_{-a}) - r_{-a}. \end{array}$$

□

Poverty, inequality and redistribution

Lemma 2 (The poor and the rich: swing voters and cut-offs)**.** *For any pair of candidates i and k there exist at most four income cut-offs: one poor swing voter $\omega^{P*}(i,k)$, one rich swing voter $\omega^{R*}(i,k)$, and the poverty lines $\omega^0(k)$ and $\omega^0(i)$.*

Proof. We start by defining the swing voter in group P, $\omega^{*P}(i,k)$, as the income level that makes a poor voter indifferent between candidates i and k; that is,

$$\psi\Big(\frac{(1 - t(r_i,p_i))\omega^{*P}(i,k) + r_i}{p_i}\Big) = \psi\Big(\frac{(1 - t(r_k,p_k))\omega^{*P}(i,k) + r_k}{p_k}\Big)$$

By monotonicity of $\psi(\cdot)$, for the previous equality to be satisfied, it must be that the arguments of function ψ are equal,

$$\begin{aligned}\frac{(1 - t(r_i,p_i))\omega^{*P}(i,k) + r_i}{p_i} &= \frac{(1 - t(r_k,p_k))\omega^{*P}(i,k) + r_k}{p_k} \\ p_i r_k - p_k r_i &= [p_k - p_i + p_i t(r_k,p_k) - p_k t(r_i,p_i)]\omega^{*P}(i,k)\end{aligned}$$

Income levels are non-negative. Then, if $p_k - p_i + p_i t(r_k,p_k) - p_k t(r_i,p_i) > (<)\, 0$, it must be that $p_i r_k - p_k r_i > (<)\, 0$. As a result, every voter in P with income

$\omega > (<)\, \omega^{*P}(i,k)$ prefers i to k. Therefore, there exists at most one swing voter in group P.
The swing voter in group R, $\omega^{*R}(i,k)$, is the income level that makes a rich voter indifferent between candidates i and k,

$$v(p_i) + (1 - t(r_i, p_i))\omega^{*R}(i,k) + r_i = v(p_k) + (1 - t(r_k, p_k))\omega^{*R}(i,k) + r_k$$
$$v(p_k) - v(p_i) + r_k - r_i = [t(r_k, p_k) - t(r_i, p_i)]\omega^{*R}(i,k)$$

If $t(r_k, p_k) - t(r_i, p_i) > (<)\, 0$, it must be that $v(p_k) - v(p_i) + r_k - r_i > (<)\, 0$. Then, every voter in R with income $\omega > (<)\, \omega^{*R}(i,k)$ prefers i to k. Therefore, there exists at most one swing voter in group R.
The expressions of the poverty lines are as follows,

$$\omega^0(k) = \frac{-v'(p_k)p_k - r_k}{1 - t_k} \leq \frac{-v'(p_i)p_i - r_i}{1 - t_i} = \omega^0(i) \tag{33}$$

□

Corollary A.2 (only in appendix) (Incentive compatibility conditions and swing voters)**.** *A two-candidate equilibrium with candidates i and k such that $\omega^0(k) \leq \omega^0(i))$ and $\omega \in (\omega^0(k), \omega^0(i))$ prefers k must satisfy,*

(*i*) *if there is one swing voter, then either*

- (1) $F(\omega^P(i,k)) = 1/2$ *or*
- (2) $F(\omega^R(i,k)) = 1/2$*;*

(*ii*) *if there are two swing voters, then either,*

- (3) $F(\omega^R(i,k)) - F(\omega^P(i,k)) = 1/2$ *or*
- (4) $F(\omega^R(i,k)) + F(\omega^P(i,k)) - F(\omega^0(k)) = 1/2$ *or*
- (5) $F(\omega^R(i,k)) + F(\omega^P(i,k)) - F(\omega^0(i)) = 1/2$ *or*
- (6) $F(\omega^R(i,k)) + F(\omega^0(k)) - F(\omega^P(i,k)) - F(\omega^0(i)) = 1/2$.

Lemma 3 (General form of policy differentials)**.** *Consider a two-candidate equilibrium with candidates i and k such that $\omega^0(k) < \omega^0(i)$. Then, for a fixed location of candidate i, the differentials of the policy components of candidate k take the form*

$$\begin{aligned} \Big(\Upsilon_a(i,k)MR_R - \Lambda_a(i,k)\Big)dr_k &= -\frac{\Lambda_a(i,k)t_k}{[1 - F(\omega^0(k))]} \times d\overline{\omega} \quad \textit{and} \\ \Big(\Upsilon_a(i,k)MR_R - \Lambda_a(i,k)\Big)dp_k &= -\frac{\Upsilon_a(i,k)t_k}{[1 - F(\omega^0(k))]} \times d\overline{\omega} \end{aligned} \tag{34}$$

where $MR_R = -v''(p_k)p_k - v'(p_k)$ is the marginal revenue derived from the demand of the regulated good by the rich, $\Upsilon_a(i,k)$ and $\Lambda_a(i,k)$ are functions of candidates' policies with $a = 1,2,3,4,5,6$ denoting the incentive compatibility conditions enumerated in Corollary 1.

Proof. We start from an equilibrium class of one swing voter. When the budget constraint is binding, a two-candidate equilibrium must satisfy the following system of equations,

$$\begin{aligned}
&F(\omega^{*P}(i,k)) = 1/2 \quad or \quad F(\omega^{*R}(i,k)) = 1/2 \\
&\omega^{*P}(i,k) = \frac{p_i r_k - p_k r_i}{p_k - p_i + p_i t_k - p_k t_i} \equiv \frac{G}{H} \\
&\omega^{*R}(i,k) = \frac{v(p_k) - v(p_i) + r_k - r_i}{t_k - t_i} \equiv \frac{I}{J} \\
&\omega^{*0}(k) = \frac{-v'(p_k)p_k - r_k}{1 - t_k} \\
&(1-t_k)W^0(k) + [1 - F(\omega^0(k))][-v'(p_k)p_k - r_k] - K - r_k + t_k\overline{\omega} = 0 \\
&W^0(k) = \int_0^{\omega^0(k)} \omega dF(\omega)
\end{aligned} \tag{35}$$

We fix the policy of candidate i. That is why we omit the budget constraint and the poverty line of i as they depend on $\mathbf{q}_i$ but not $\mathbf{q}_k$. Taking differentials of the budget constraint in 35,

$$\begin{aligned}
&f(\omega^0(k))[(1-t_k)\omega^0(k) + r_k + v'(p_k)p_k] \times \left(\frac{\partial\omega^0(k)}{\partial t_k}dt_k + \frac{\partial\omega^0(k)}{\partial r_k}dr_k + \frac{\partial\omega^0(k)}{\partial p_k}dp_k\right) \\
&- W^0(k)dt_k + [1 - F(\omega^0(k))][-v''(p_k)p_k - v'(p_k)]dp_k \\
&- [1 - F(\omega^0(k))]dr_k - dK + \overline{\omega}dt_k + t_k d\overline{\omega} = 0
\end{aligned}$$

The introduction of the definition of $\omega^0(k)$ results in $(1-t_k)\omega^0(k) + r_k + v'(p_k)p_k = 0$. Then, the differentials of the budget constraint, $\omega^0(k)$, $\omega^{*P}(i,k)$ and $\omega^{*R}(i,k)$ are given by,

$$\begin{aligned}
&d\omega^{*P}(i,k) = [H(p_i dr_k - r_i dp_k) - G((1-t_i)dp_k + p_i dt_k)] \times H^{-2} \\
&d\omega^{*R}(i,k) = [Jv'(p_k)dp_k - Idt_k] \times I^{-2} \\
&d\omega^0(k) = \{(1-t_k)(MR_R dp_k - dr_k) - [-v'(p_k)p_k - r_k]dt_k\} \times (1-t_k)^{-2} \\
&- W^0(k)dt_k + [1 - F(\omega^0(k))]MR_R dp_k \\
&- [1 - F(\omega^0(k))]dr_k - dK + \overline{\omega}dt_k + t_k d\overline{\omega} = 0
\end{aligned}$$

where $MR_R = -v''(p_k)p_k - v'(p_k)$ is the marginal revenue from the provision of regulated good to the rich.

Now consider the case where the only swing voter is poor; i.e., $F(\omega^{*P}(i,k)) = 1/2$. Then, for a given policy of candidate i, a change in candidate k's policy must satisfy $f[\omega^{*P}(i,k)]d\omega^{*P}(i,k) = 0$ for still being at equilibrium. If $\omega^{*P}(i,k) \in supp(f)$, then $f[\omega^{*P}(i,k)] > 0$, and therefore, $d\omega^{*P}(i,k) = 0$. If we further consider $dt_k = dK = 0$, we have

$$\begin{aligned} d\omega^{*P}(i,k) &= Hp_i dr_k - [Hr_i + G(1-t_i)]dp_k = 0 \\ dp_k &= -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{36}$$

Let $\Upsilon_1 \equiv Hp_i$ and $\Lambda_1 \equiv [Hr_i + G(1-t_i)]$. Then, system 36 is equivalent to

$$\begin{aligned} &\Upsilon_1 dr_k - \Lambda_1 dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{37}$$

The solution of this system is given by

$$\begin{aligned} \Big(\Upsilon_1 MR_R - \Lambda_1\Big)dr_k &= -\frac{\Lambda_1 t_k}{[1-F(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_1 MR_R - \Lambda_1\Big)dp_k &= -\frac{\Upsilon_1 t_k}{[1-F(\omega^0(k))]} \times d\varpi \end{aligned} \tag{38}$$

Consider the case where the only swing voter is rich; i.e., $F(\omega^{*R}(i,k)) = 1/2$. Then, for a given policy of candidate i, a change in candidate k's policy must satisfy $f[\omega^{*R}(i,k)]d\omega^{*R}(i,k) = 0$ for still being at equilibrium. If $\omega^{*R}(i,k) \in supp(f)$, then $f[\omega^{*R}(i,k)] > 0$, and therefore, $d\omega^{*R}(i,k) = 0$. If we further consider $dt_k = dK = 0$, we have

$$\begin{aligned} d\omega^{*R}(i,k) &= dr_k - [-v'(p_k)]dp_k = 0 \\ dp_k &= -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{39}$$

Let $\Upsilon_2 \equiv 1$ and $\Lambda_2 \equiv -v'(p_k)$. Then, system 39 is equivalent to

$$\begin{aligned} &\Upsilon_2 dr_k - \Lambda_2 dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{40}$$

The solution of this system is given by

$$\begin{aligned} \Big(\Upsilon_2 MR_R - \Lambda_2\Big)dr_k &= -\frac{\Lambda_2 t_k}{[1-F(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_2 MR_R - \Lambda_2\Big)dp_k &= -\frac{\Upsilon_2 t_k}{[1-F(\omega^0(k))]} \times d\varpi \end{aligned} \tag{41}$$

When there are two swing voters that are also the only cut-offs, the equilibrium satisfies $F(\omega^{*R}(i,k)) - F(\omega^{*P}(i,k)) = 1/2$. Then, for a given policy of candidate i, a change in candidate k's policy must satisfy $f[\omega^{*R}(i,k)]d\omega^{*R}(i,k) - f[\omega^{*P}(i,k)]d\omega^{*P}(i,k) = 0$ for still being at equilibrium. Then,

$$\begin{aligned} &\Big[\frac{f(\omega^R)}{J} - \frac{f(\omega^P)}{H}p_i\Big]dr_k - \Big[-\frac{f(\omega^P)}{H}\omega^P(1-t_i) - \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k)\Big]dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{42}$$

Let

$$\begin{aligned} \Upsilon_3 &\equiv \frac{f(\omega^R)}{J} - \frac{f(\omega^P)}{H}p_i \quad \text{and} \\ \Lambda_3 &\equiv -\frac{f(\omega^P)}{H}\omega^P(1-t_i) - \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) \end{aligned}$$

Then, system 42 is equivalent to

$$\begin{aligned} &\Upsilon_3 dr_k - \Lambda_3 dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{43}$$

The solution of this system is given by

$$\begin{aligned} \Big(\Upsilon_3 MR_R - \Lambda_3\Big)dr_k &= -\frac{\Lambda_3 t_k}{[1-F(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_3 MR_R - \Lambda_3\Big)dp_k &= -\frac{\Upsilon_3 t_k}{[1-F(\omega^0(k))]} \times d\varpi \end{aligned} \tag{44}$$

When there are two swing voters and one additional cut-off at $\omega^0(k)$, the equilibrium satisfies $F[\omega^{*R}(i,k)] + F[\omega^{*P}(i,k)] - F[\omega^0(k)] = 1/2$. Then, for a given policy of candidate i, a change in candidate k's policy must satisfy $f[\omega^{*R}(i,k)]d\omega^{*R}(i,k) + f[\omega^{*P}(i,k)]d\omega^{*P}(i,k) - f[\omega^0(k)]d\omega^0(k) = 0$ for still being at equilibrium. Then,

$$\begin{aligned} &\Big[\frac{f(\omega^R)}{J} + \frac{f(\omega^P)}{H}p_i + \frac{f(\omega^0)}{1-t_k}\Big]dr_k \\ &- \Big[\frac{f(\omega^P)}{H}\omega^P(1-t_i) + \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) + MR_R\frac{f(\omega^0)}{1-t_k}\Big]dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{45}$$

Let

$$\Upsilon_4 \equiv \frac{f(\omega^R)}{J} + \frac{f(\omega^P)}{H}p_i + \frac{f(\omega^0)}{1-t_k} \quad \text{and}$$
$$\Lambda_4 \equiv \frac{f(\omega^P)}{H}\omega^P(1-t_i) + \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) + MR_R\frac{f(\omega^0)}{1-t_k}$$

Then, system 45 is equivalent to

$$\begin{aligned} &\Upsilon_4 dr_k - \Lambda_4 dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{46}$$

The solution of this system is given by

$$\begin{aligned} \Big(\Upsilon_4 MR_R - \Lambda_4\Big)dr_k &= -\frac{\Lambda_4 t_k}{[1-F(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_4 MR_R - \Lambda_4\Big)dp_k &= -\frac{\Upsilon_4 t_k}{[1-F(\omega^0(k))]} \times d\varpi \end{aligned} \tag{47}$$

When there are two swing voters and one additional cut-off at $\omega^0(i)$, the equilibrium satisfies $F[\omega^{*R}(i,k)] + F[\omega^{*P}(i,k)] - F[\omega^0(i)] = 1/2$.
Then, for a given policy of candidate i, a change in candidate k's policy must satisfy $f[\omega^{*R}(i,k)]d\omega^{*R}(i,k) + f[\omega^{*P}(i,k)]d\omega^{*P}(i,k)$ since $d\omega^0(i) = 0$.
Then,

$$\begin{aligned} &\Big[\frac{f(\omega^R)}{J} + \frac{f(\omega^P)}{H}p_i\Big]dr_k - \Big[\frac{f(\omega^P)}{H}\omega^P(1-t_i) + \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k)\Big]dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{48}$$

Let

$$\Upsilon_5 \equiv \frac{f(\omega^R)}{J} + \frac{f(\omega^P)}{H}p_i \quad \text{and}$$
$$\Lambda_5 \equiv \frac{f(\omega^P)}{H}\omega^P(1-t_i) + \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k)$$

Then, system 48 is equivalent to

$$\begin{aligned} &\Upsilon_5 dr_k - \Lambda_5 dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k \end{aligned} \tag{49}$$

The solution of this system is given by

$$\begin{aligned}\Big(\Upsilon_5 MR_R - \Lambda_5\Big) dr_k &= -\frac{\Lambda_5 t_k}{[1-F(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_5 MR_R - \Lambda_5\Big) dp_k &= -\frac{\Upsilon_5 t_k}{[1-F(\omega^0(k))]} \times d\varpi\end{aligned} \tag{50}$$

When there are two swing voters and two additional cut-offs at $\omega^0(i)$ and $\omega^0(k)$, the equilibrium satisfies $F[\omega^{*R}(i,k)] - F[\omega^{*P}(i,k)] + F[\omega^0(k)] - F[\omega^0(i)] = 1/2$. Then, for a given policy of candidate i, a change in candidate k's policy must satisfy $f[\omega^{*R}(i,k)]d\omega^{*R}(i,k) - f[\omega^{*P}(i,k)]d\omega^{*P}(i,k) + f[\omega^0(k)]d\omega^0(k) = 0$ since $d\omega^0(i) = 0$. Then,

$$\begin{aligned}&\Big[\frac{f(\omega^R)}{J} - \frac{f(\omega^P)}{H}p_i - \frac{f(\omega^0)}{1-t_k}\Big]dr_k \\ &-\Big[-\frac{f(\omega^P)}{H}\omega^P(1-t_i) - \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) - MR_R\frac{f(\omega^0)}{1-t_k}\Big]dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k\end{aligned} \tag{51}$$

Let

$$\begin{aligned}\Upsilon_6 &\equiv \frac{f(\omega^R)}{J} - \frac{f(\omega^P)}{H}p_i - \frac{f(\omega^0)}{1-t_k} \quad \text{and} \\ \Lambda_6 &\equiv -\frac{f(\omega^P)}{H}\omega^P(1-t_i) - \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) - MR_R\frac{f(\omega^0)}{1-t_k}\end{aligned}$$

Then, system 51 is equivalent to

$$\begin{aligned}&\Upsilon_6 dr_k - \Lambda_6 dp_k = 0 \\ &dp_k = -\frac{t_k}{[1-F(\omega^0(k))]MR_R}d\varpi + \frac{1}{MR_R}dr_k\end{aligned} \tag{52}$$

The solution of this system is given by

$$\begin{aligned}\Big(\Upsilon_6 MR_R - \Lambda_6\Big) dr_k &= -\frac{\Lambda_6 t_k}{[1-F(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_6 MR_R - \Lambda_6\Big) dp_k &= -\frac{\Upsilon_6 t_k}{[1-F(\omega^0(k))]} \times d\varpi\end{aligned} \tag{53}$$

□

Proposition 2 (One-cut-off equilibrium and policy components). *Consider a one-cut-off equilibrium with candidates i and k. Then, for any continuous change in the income distribution such that the new distribution F' satisfies $F(x) < F'(x)$ with $x = \omega^{*b}(i,k)$ and $b = P, R$, it must be that;*

(a)

$$
\begin{aligned}
dr_k &< L\Big(\Upsilon_1, \Lambda_1, F'[\omega^0(k)], MR_R\Big) \times d\overline{\omega} \ \ \textit{and} \\
dp_k &< M\Big(\Upsilon_1, \Lambda_1, F'[\omega^0(k)], MR_R\Big) \times d\overline{\omega}
\end{aligned}
\tag{54}
$$

if the cut-off is $\omega^{*P}(i,k)$ *and* $\frac{\omega^{*P}(i,k)(1-t_i)+r_i}{p_i} > MR_R$*, or;*

(b)

$$
\begin{aligned}
dr_k &> L\Big(\Upsilon_2, \Lambda_2, F'[\omega^0(k)], MR_R\Big) \times d\overline{\omega} \ \ \textit{and} \\
dp_k &> M\Big(\Upsilon_2, \Lambda_2, F'[\omega^0(k)], MR_R\Big) \times d\overline{\omega}
\end{aligned}
\tag{55}
$$

if the cut-off is $\omega^{*R}(i,k)$*;*

with $L(\cdot), M(\cdot) > 0$*.*

Proof. **Part (a)**. Consider the case where the only swing voter is poor; i.e., $F(\omega^{*P}(i,k)) = 1/2$. If the distribution F suffers a continuous change to F' such that $F < F'$ at $\omega^{*P}(i,k)$, then $F'(\omega^{*P}(i,k)) > 1/2$. As a consequence, in order to re-establish the equilibrium it must be that $d\omega^{*P}(i,k) < 0$. From Lemma 3, this is equivalent to,

$$
\begin{aligned}
\Big(\Upsilon_1 MR_R - \Lambda_1\Big) dr_k &< -\frac{\Lambda_1 t_k}{[1 - F'(\omega^0(k))]} \times d\overline{\omega} \\
\Big(\Upsilon_1 MR_R - \Lambda_1\Big) dp_k &< -\frac{\Upsilon_1 t_k}{[1 - F'(\omega^0(k))]} \times d\overline{\omega}
\end{aligned}
\tag{56}
$$

with $\Upsilon_1 = Hp_i$ and $\Lambda_1 = [Hr_i + G(1-t_i)]$. By Lemma 2, $H, G < 0$ since all voters ω such that $\omega > \omega^{*P}$ must vote for k. If $\Upsilon_1 MR_R - \Lambda_1 > 0$, then

$$
\begin{aligned}
dr_k &< -\frac{\Lambda_1 t_k}{[1 - F'(\omega^0(k))]} \times \frac{1}{\Upsilon_1 MR_R - \Lambda_1} \times d\overline{\omega} \\
dp_k &< -\frac{\Upsilon_1 t_k}{[1 - F'(\omega^0(k))]} \times \frac{1}{\Upsilon_1 MR_R - \Lambda_1} \times d\overline{\omega}
\end{aligned}
\tag{57}
$$

And $\Upsilon_1 MR_R - \Lambda_1 > 0$ if and only if

$$
\begin{aligned}
Hp_i MR_R - Hr_i - G(1-t_i) > 0 \\
\frac{\omega^{*P}(i,k)(1-t_i) + r_i}{p_i} > MR_R
\end{aligned}
\tag{58}
$$

where we use $\omega^{*P}(i,k) \equiv \frac{G}{H}$ and $G, H < 0$.

Part (b). Consider the case where the only swing voter is rich; i.e.,

$F(\omega^{*R}(i,k)) = 1/2$. If the distribution F suffers a continuous change to F' such that $F < F'$ at $\omega^{*R}(i,k)$, then $F'(\omega^{*R}(i,k)) > 1/2$. As a consequence, in order to re-establish the equilibrium it must be that $d\omega^{*R}(i,k) < 0$. From Lemma 3, this is equivalent to,

$$\begin{aligned} \Big(\Upsilon_2 MR_R - \Lambda_2\Big) dr_k &< -\frac{\Lambda_1 t_k}{[1-F'(\omega^0(k))]} \times d\varpi \\ \Big(\Upsilon_2 MR_R - \Lambda_2\Big) dp_k &< -\frac{\Upsilon_2 t_k}{[1-F'(\omega^0(k))]} \times d\varpi \end{aligned} \tag{59}$$

with $\Upsilon_2 = 1$ and $\Lambda_2 = -v'(p_k)$. If $\Upsilon_2 MR_R - \Lambda_2 < 0$, then

$$\begin{aligned} dr_k &> -\frac{\Lambda_2 t_k}{[1-F'(\omega^0(k))]} \times \frac{1}{\Upsilon_2 MR_R - \Lambda_2} \times d\varpi \\ dp_k &> -\frac{\Upsilon_2 t_k}{[1-F'(\omega^0(k))]} \times \frac{1}{\Upsilon_2 MR_R - \Lambda_2} \times d\varpi \end{aligned} \tag{60}$$

But it is always the case that $\Upsilon_2 MR_R - \Lambda_2 < 0$ since $\Upsilon_2 MR_R - \Lambda_2 = MR_R + v'(p_k) = -v''(p_k)p_k < 0$. It is only pending to define,

$$\begin{aligned} L &\equiv -\frac{\Lambda_a t_k}{[1-F'(\omega^0(k))]} \times \frac{1}{\Upsilon_a MR_R - \Lambda_a} \\ M &\equiv -\frac{\Upsilon_a t_k}{[1-F'(\omega^0(k))]} \times \frac{1}{\Upsilon_a MR_R - \Lambda_a} \end{aligned} \tag{61}$$

with $a = 1, 2$. □

Proposition 3 (Multiple-cut-offs equilibria and policy components)**.** *In any multiple-cut-off equilibrium with candidates i and k where $\frac{\omega^{*P}(i,k)+r_i}{p_i} > MR_R$,*

$$\begin{aligned} dr_k &= L\Big(\Upsilon_a, \Lambda_a, F[\omega^0(k)], MR_R\Big) \times d\varpi \;\; \textit{and} \\ dp_k &= M\Big(\Upsilon_a, \Lambda_a, F[\omega^0(k)], MR_R\Big) \times d\varpi \end{aligned} \tag{62}$$

where $L(\cdot), M(\cdot) > 0$, and $a = 3, 4, 5, 6$ denotes the incentive compatibility conditions enumerated in Corollary 1.

Proof. We start defining $L\Big(\Upsilon_a, \Lambda_a, F[\omega^0(k)], MR_R\Big)$ and $M\Big(\Upsilon_a, \Lambda_a, F[\omega^0(k)], MR_R\Big)$ as following,

$$\begin{aligned} L &\equiv -\frac{\Lambda_a t_k}{[1-F(\omega^0(k))]} \times \frac{1}{\Upsilon_a MR_R - \Lambda_a} \\ M &\equiv -\frac{\Upsilon_a t_k}{[1-F(\omega^0(k))]} \times \frac{1}{\Upsilon_a MR_R - \Lambda_a} \end{aligned} \tag{63}$$

We want to prove that $\frac{\omega^{*P}(i,k)+r_i}{p_i} > MR_R$ is a sufficient condition for both $L(\cdot) > 0$ and $M(\cdot) > 0$, for $a = 3, ..., 6$. Or equivalently, if $\Upsilon_a MR_R - \Lambda_a > (<)0$, then $\Upsilon_a, \Lambda_a < (>)0$.
Consider the case where $a = 3$. By Lemma 3,

$$\Upsilon_3 \equiv \frac{f(\omega^R)}{J} - \frac{f(\omega^P)}{H}p_i \quad \text{and}$$
$$\Lambda_3 \equiv -\frac{f(\omega^P)}{H}\omega^P(1-t_i) - \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k)$$

Since $H < 0$ and $J > 0$, then $\Upsilon_3 > 0$ and $\Lambda_3 > 0$. Furthermore, $\Upsilon_3 MR_R - \Lambda_3 < 0$ if and only if

$$\frac{f(\omega^R)}{J}(-v''(p_k)p_k) - \frac{f(\omega^P)}{H}\Big[p_i MR_R - \omega^P(1-t_i) - r_i\Big] < 0 \tag{64}$$

A sufficient condition for 64 is that $\frac{\omega^{*P}(i,k)+r_i}{p_i} > MR_R$.
Consider the case where $a = 4$. By Lemma 3,

$$\Upsilon_4 \equiv \frac{f(\omega^R)}{J} + \frac{f(\omega^P)}{H}p_i + \frac{f(\omega^0)}{1-t_k} \quad \text{and}$$
$$\Lambda_4 \equiv \frac{f(\omega^P)}{H}\omega^P(1-t_i) + \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) + MR_R\frac{f(\omega^0)}{1-t_k}$$

Since $H > 0$ and $J > 0$, then $\Upsilon_4 > 0$ and $\Lambda_4 > 0$. Furthermore, $\Upsilon_4 MR_R - \Lambda_4 < 0$ if and only if

$$\frac{f(\omega^R)}{J}(-v''(p_k)p_k) - \frac{f(\omega^P)}{H}\Big[-p_i MR_R + \omega^P(1-t_i) + r_i\Big] < 0 \tag{65}$$

A sufficient condition for 65 is that $\frac{\omega^{*P}(i,k)+r_i}{p_i} > MR_R$.
Consider the case where $a = 5$. By Lemma 3,

$$\Upsilon_5 \equiv \frac{f(\omega^R)}{J} + \frac{f(\omega^P)}{H}p_i \quad \text{and}$$
$$\Lambda_5 \equiv \frac{f(\omega^P)}{H}\omega^P(1-t_i) + \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k)$$

Since $H < 0$ and $J < 0$, then $\Upsilon_5 < 0$ and $\Lambda_5 < 0$. Furthermore, $\Upsilon_5 < MR_R - \Lambda_5 > 0$ if and only if

$$\frac{f(\omega^R)}{J}(-v''(p_k)p_k) + \frac{f(\omega^P)}{H}\Big[p_i MR_R - \omega^P(1-t_i) + r_i\Big] > 0 \tag{66}$$

A sufficient condition for 66 is that $\frac{\omega^{*P}(i,k)+r_i}{p_i} > MR_R$.
By Lemma 3,

$$\Upsilon_6 \equiv \frac{f(\omega^R)}{J} - \frac{f(\omega^P)}{H}p_i - \frac{f(\omega^0)}{1-t_k} \quad \text{and}$$
$$\Lambda_6 \equiv -\frac{f(\omega^P)}{H}\omega^P(1-t_i) - \frac{f(\omega^P)}{H}r_i - \frac{f(\omega^R)}{J}v'(p_k) - MR_R\frac{f(\omega^0)}{1-t_k}$$

Since $H > 0$ and $J < 0$, then $\Upsilon_6 < 0$ and $\Lambda_6 < 0$. Furthermore, $\Upsilon_6 MR_R - \Lambda_6 > 0$ if and only if

$$\frac{f(\omega^R)}{J}(-v''(p_k)p_k) - \frac{f(\omega^P)}{H}\Big[p_i MR_R - \omega^P(1-t_i) + r_i\Big] > 0 \tag{67}$$

A sufficient condition for 67 is that $\frac{\omega^{*P}(i,k)+r_i}{p_i} > MR_R$. □